MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

CONTAINING

PAPERS, ABSTRACTS OF PAPERS, AND REPORTS OF THE PROCEEDINGS OF THE SOCIETY

FROM NOVEMBER 1936 TO OCTOBER 1937

VOL. 97

Published and Sold by the
ROYAL ASTRONOMICAL SOCIETY
BURLINGTON HOUSE
LONDON, W. 1

1937

Printed in Great Britain by
Neill & Co., Ltd., Edinburgh.

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Vol. 97 No. 1

MEETING OF 1936 NOVEMBER 13

Mr. J. H. Reynolds, President, in the Chair.

The President announced the death of Dr. W. J. S. Lockyer, a member of Council; and also the deaths of three Associates of the Society—Professor Paul Stroobant, Dr. Johannes Hartmann and Dr. Friedrich Küstner.

The Treasurer announced that the whole of the Herbert Spencer Bequest had now been received, amounting in all to the sum of £1241. He drew attention to the improvements effected in the Meeting Room since the last meeting, which were being paid for from the Bequest.

The Treasurer further announced that the late Captain J. J. L. Goodridge, F.R.A.S., had bequeathed to the Society the sum of £50, free of legacy duty.

The President announced that the library would be open until 21 h. on Wednesday evenings throughout the session.

The President announced that the Library Committee had drawn up a list of books which are no longer to be retained in the library and which are to be offered gratis to Fellows. This list will be laid on the library table for the inspection of Fellows, who are requested to apply to the Assistant Secretary for any books which they may select.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

M. Akhtar, M.Sc., Imperial College of Science, S. Kensington (proposed by Herbert Dingle) ;

G. L. Clark, B.A., 164 Gilbert Road, Cambridge (proposed by W. M. Smart) ;

Mrs. Rosina Dafter, Flower Road, Northgate, Brisbane, Queensland, Australia (proposed by R. B. Bousfield) ;

William Samuel Davenport, Commander, R.D., F.R.Met.Soc., R.N.R., Questings, Arundel Road, Peacehaven, Sussex (proposed by H. Spencer Jones) ;

J. Fleming, B.A., Trinity Hall, Cambridge (proposed by W. M. Smart) ;

Sydney I. Luck, 26 Wellington Square, Chelsea, S.W. 3 (proposed by J. A. Carroll) ;

Eugene Fairfield MacPike, 135 East Eleventh Place, Chicago, Ill., U.S.A. (proposed by W. M. H. Greaves) ;

Kenneth Mitchell, Ph.D., B.A., Assistant Lecturer in Mathematics, The University, Leeds 2 (proposed by S. Brodetsky) ;

Charles Mowat, Leith Nautical College, Edinburgh 6 (proposed by J. Storey) ;

Thornton Leigh Page, University Observatory, Oxford (proposed by H. H. Plaskett) ;

John Fraser Paterson, Box 151A, Broken Hill, N.S.W., Australia (proposed by W. F. Gale) ;

C. F. Retter, M.A., B.Sc., 90 Blake Road, New Southgate, London, N. 11 (proposed by G. H. Gilbert) ;

The Dowager Lady Saltoun, 25 Gloucester Place, London, W. 1 (proposed by W. M. Smart) ;

George Whalley Wakeford, Lieutenant, R.N.R., Dept. of Navigation, University College, Southampton (proposed by W. H. Coombs) ; and

Oswald Martin Watts, A.I.N.A., 286 Leigham Court Road, Streatham, London, S.E. 16 (proposed by W. F. Doak).

Two hundred and fifty-three presents were announced as having been received since the last meeting, including :—

Ethel F. Bellamy, Index Catalogue of Epicentres for 1913–1930 ; a geographical index to the International Seismological Summary (presented by the author) ;

Sir Arthur Eddington, Relativity Theory of Protons and Electrons (presented by the author);

Harvard College, Cambridge (Mass.) Branch of the Astronomical Society, Records of proceedings, 1854–57 (presented by Dr. Shapley);

Edwin Hubble, The Realm of the Nebulæ (presented by the author);

C. E. K. Mees, Photography (presented by the author).

The reports of the expeditions sent to observe the eclipse of 1936 June 19 were presented.

ON THE FIGURES OF THE EARTH AND MOON.

Harold Jeffreys, D.Sc., F.R.S.

1. In an important series of papers de Sitter has developed the theory of the figure of the Earth to the second order of small quantities, in the first place using only the theory of the external field, and in the second a more accurate form of Darwin's theory, assuming hydrostatic stress in the interior and computing the ellipticity and related constants from the precessional constant.* He has also compared the results with the motion of the Moon's node and perigee. The latter are well known from observation, and the greater parts are explained by Brown's theory, but there is a balance which depends on the figures of the Earth and Moon. E. W. Brown gives expressions for the effects of these.† Previous writers had usually attributed the whole of this to the figure of the Earth and obtained ellipticities which appeared to be in disagreement with determinations from gravity. De Sitter takes the part due to the Earth as calculable from his theory, and is left with a small balance which he attributes to the Moon's ellipticities. The results are on the whole plausible, and the apparent uncertainty of the ellipticity as found from the precessional constant is only about a quarter of that given by the best analysis of gravity observations. Unfortunately, however, the assumption of hydrostatic stress in the interior does not agree with other data. It implies that there should be no inequalities of gravity over the Earth's surface except the main ellipticity effect and local ones agreeing with the hypothesis of isostasy. According to de Sitter's solution, the probable error of the difference between gravity at the poles and the equator is only 1 milligal (0·001 cm./sec.^{2}). But Heiskanen's

* *K. Akad. Wetens. Amst.*, **17**, 1291–1321, 1915; *Bull. Astr. Soc. Neth.*, **2**, No. 55, 97–108, 1924; **4**, No. 129, 57–61, 1927. These will be referred to as Papers I, II and III.

† *Mems. R.A.S.*, **59**, 80, 1915.

ellipticity of the equator alone gives a variation of 38 milligals at the extremes, and there may be comparable anomalies of the third and fourth degrees, which will affect the coefficient of the main term if they are not explicitly calculated. These inequalities are inconsistent with de Sitter's basic hypothesis, and we should be prepared to find that the ellipticity found by him may be wrong by 20 times its apparent uncertainty. The probable error of Heiskanen's formula corresponds to a variation of 4 milligals in the range of the main term; but this also is probably an underestimate, since the residuals are correlated at points up to a considerable distance apart.

De Sitter's theory, however, has the great merit that it is a consistent whole, and can be used for purposes of comparison, while none of the other formulæ have been tested against the motion of the Moon. It is affected by one small error, which de Sitter pointed out to me in a letter dated 1932 April 22, the relativity correction having been applied with the wrong sign; he gave me the corrected values for some of the constants. In this paper I evaluate some others.

Geophysical information can now be applied to determine the ratio C'/Mb'^2 for the Moon more accurately than de Sitter's theory can give it. If we write, with de Sitter,

$$\mathcal{J}=\frac{3}{2}\frac{2C-A-B}{Mb^2}, \quad \mathcal{J}'=\frac{3}{2}\frac{2C'-A'-B'}{2M'b'^2}, \quad K'=\frac{3}{2}\frac{B'-A'}{M'b'^2},$$

$$H=\frac{C-A}{C}, \quad \beta=\frac{C'-A'}{C'}, \quad \alpha=\frac{C'-B'}{C'}, \quad q=\frac{3}{2}\frac{C}{Mb^2}, \quad g'=\frac{3}{2}\frac{C'}{M'b'^2}, \quad (1)$$

unaccented letters referring to the Earth and accented ones to the Moon, we have the relations:

$$g'\beta=\mathcal{J}'+\tfrac{1}{2}K'; \quad g'\alpha=\mathcal{J}'-\tfrac{1}{2}K'; \quad f=\alpha/\beta=(2\mathcal{J}'-K')/(2\mathcal{J}'+K'). \quad (2)$$

The axes are as follows: A' is the moment of inertia about the axis pointing to the Earth, while C' corresponds to the axis of rotation and B' to that in the direction of motion. This selection is the same as that used by Stratton, Brown and myself. De Sitter does not define the axes explicitly, but his notation corresponds exactly to Brown's, $\mathcal{J}'$ and K' being simple multiples of Brown's μ' and μ''.

The motions of the node and perigee depend on $\mathcal{J}$, $\mathcal{J}'$ and K', and the inclination of the Moon's equator to the ecliptic depends mainly on β. Thus if g' is known we have a soluble problem. Then $\mathcal{J}$ gives determinations of the ellipticity of the Earth and the main ellipticity term in gravity from the theory of the Earth's external field alone, without reference to any hypothesis about the state of the interior. This has been done in the present paper.

Other information about $\mathcal{J}$ is provided by a term in the Moon's latitude mentioned but not used by de Sitter, and about $\beta-\alpha$ by the librations in longitude. These are connected with the parameter f, concerning which there is discordant evidence. Astronomers relying on Schlüter's observations have usually found f about 0·5, but Hayn gets 0·75. The theoretical

value for a satellite that solidified when it had already come to turn a constant face to its primary is 0·25 ; in other words, the ellipticity of the visible disk should be a third of that in the plane of the equator. In my work * I seem to have misread Hayn and thought that his f was $(B-A)/(C-A)$; when this is corrected Hayn's observations no longer support this hypothesis. De Sitter remarks that the amplitude of the libration subtends only 1″·4 at the Earth, but I think this should be 0″·7 ; in any case it appears that the difficulties of measuring it have so far been too great. Stability conditions ensure that f must be between 0 and 1 ; we have the following comparison between f and $K'/2\mathfrak{J}'$:—

f	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$K'/2\mathfrak{J}'$	0	$\frac{1}{7}$	$\frac{1}{3}$	$\frac{3}{5}$	1

De Sitter's formula for gravity (Paper III) is

$$g=(978\cdot052\pm\cdot002)\{1+(0\cdot0052884\pm\cdot0000011)\sin^2\phi-0\cdot0000075\sin^2 2\phi\}. \quad (3)$$

Heiskanen's formula, allowing for the ellipticity of the equator, is

$$g=(978\cdot049\pm\cdot003)\{1+(0\cdot005293\pm\cdot000004)\sin^2\phi-0\cdot000007\sin^2 2\phi +(0\cdot000019\pm\cdot000003)\cos^2\phi\cos 2(\lambda-0^\circ\pm5^\circ)\}. \quad (4)$$

His later formula, omitting this term, is

$$g=978\cdot049(1+0\cdot005289\sin^2\phi-0\cdot000007\sin^2 2\phi), \quad (5)$$

and the international formula due to Cassinis is

$$g=978\cdot049(1+0\cdot0052884\sin^2\phi-0\cdot0000059\sin^2 2\phi). \quad (6)$$

In Heiskanen's two formulæ the term in $\sin^2 2\phi$ rests on a second-order theory such as those of Darwin and de Sitter, which make the form of a level surface slightly different from an ellipsoid. In the international formula it is adjusted to make the level surface an ellipsoid of revolution. It does not rest on observational evidence, but if we keep second-order terms at all they will in general appear in both g and the radius vector, and some theoretical condition is needed to make them consistent. If the potential contains an independent term in P_4 it can be allowed for in the figure by the first-order theory of Stokes.

De Sitter's theory appeared at first (Paper I) to lead to a difficulty with regard to the figure of the Moon. The number g' was found to be 0·70 ± 0·07 ; † it was expected to agree with q, which is the corresponding number for the Earth, and is about 0·50. For a homogeneous sphere q would be 0·60. Taken at its face value $q=0\cdot70$ would mean that the Moon's density decreases inwards, and in any case it suggested that the Moon is less condensed to the centre than the Earth. In Paper III de Sitter gets 0·65 ± 0·13 for g'. Actually the mean density of the Moon is very near

* *Mems. R.A.S.*, **60**, 216, 1915.

† De Sitter gives mean square errors in Paper I, probable errors in Papers II and III. My own usual practice is to give mean square errors, but as continual reference has to be made to de Sitter's papers I use probable errors in this paper.

what has been adopted for the "lower layer" of the Earth, below about 40 km. depth, and it is probable that the Moon is nearly homogeneous. Thus it becomes possible to estimate g' with considerable accuracy. It is definitely larger than q, and is nearly 0·6.

2. With regard to the relativity correction, the following table gives in the first column the values from Paper III, and the second the corrected ones from de Sitter's letter; so far as I know these have not been published. Probable errors remain practically as in the paper.

$p =$	50″·2486 (observed)	
$p_0 =$	50″·3523	50″·3910
$H =$	0·0032770	0·0032794
$q =$	0·50009	0·5005
$\mathcal{J} =$	0·0016388	0·0016412
$1/e =$	296·96	296·75
$e =$	0·0033675	0·0033698

The change will also reduce the coefficient of $\sin^2\phi$ in g/g_0 from 0·0052884 to 0·0052860. The correction to the precession is due to A. D. Fokker; * an alternative discussion is given by Eddington.† It is shown that dynamical phenomena near the Earth will be affected by the Earth's motion and the Sun's field in such a way that the resultant motion referred to non-rotating axes is the same as the Newtonian motion referred to axes rotating with angular velocity −0″·0194 per year. Thus the correction gives an angular velocity +0″·0194 in the plane of the ecliptic to all motions near the Earth, in addition to any effects calculable by Newtonian dynamics. The observed precession being −50″·2486, we have −50″·2680 due to Newtonian forces. Subtracting also the planetary precession +0″·1228, we have 50″·3908, which is practically de Sitter's revised value. He remarks in his letter that he has also applied a small correction to the planetary precession.

The change in e gives an increase of 0″·003 in the calculated lunar parallax, giving 3422″·522 ± 0·009. The calculated motions of the lunar perigee and node due to the Earth's figure are proportional to $\mathcal{J}$, and are increased from 6″·386 and 5″·977 to 6″·396 and 5″·986 respectively. But the outstanding probable errors of the difference between observation and theory are estimated at 0″·06 and 0″·04, so that the change of the calculated effect of the Earth's figure makes no perceptible difference in the contribution from the figure of the Moon. The calculated nutation coefficient is now 9″·2175 ± 0·0018, while the observed value is 9″·2075 ± 0·002.‡ The difference is 3·7 times its probable error, or 2·5 times its standard error, and doubtfully significant.

On the whole, therefore, while de Sitter's theory rests on a doubtful foundation, it does not lead to obviously wrong results within this range of data.

* *Proc. R. Acad. Sci. Amst.*, **23**, 729–738, 1921.

† *Mathematical Theory of Relativity*, pp. 95–99, 1923.

‡ J. Jackson, *M.N.*, **90**, 741, 1930.

3. To proceed in the reverse direction, we must begin with the quantity g'. For the Earth there is evidence that the density 40 km. down is about 3·3 gm./cm.^{3}, which is near the mean density of the Moon. We assume that the material is the same. The velocity of longitudinal waves is about 7·8 km./sec., that of distortional waves 4·4 km./sec. These give for the ratio of the bulk-modulus to the density

$$k/\rho = \alpha^2 - \tfrac{4}{3}\beta^2 = 3{\cdot}5 \times 10^{11} \text{ cm.}^2/\text{sec.}^2. \quad (1)$$

Surface gravity is 162 gal ; assuming homogeneity as a first approximation, we therefore find the pressure

$$p = \frac{1}{2}\frac{\rho g_0}{b'}(b'^2 - r^2), \quad (2)$$

and the increase of density due to pressure is

$$\rho p/k = 1{\cdot}4 \times 10^{-12}\frac{\rho g_0}{b'}(b'^2 - r^2) = 0{\cdot}13(1 - r^2/b'^2). \quad (3)$$

Thus pressure alone will not increase the density at the centre by more than 0·13 gm./cm.^{3}, and as far as this goes homogeneity may be a good approximation ; we can take the surface density to be ρ_0, and then

$$\rho = \rho_0 + 0{\cdot}13(1 - r^2/b'^2). \quad (4)$$

Then for the mean density

$$\bar{\rho} = \rho_0 + 0{\cdot}052 = 3{\cdot}33. \quad (5)$$

Thus the surface density is 3·28 and the central density 3·41. The moment of inertia is the same as for a homogeneous body of density $\rho_0 + 0{\cdot}037$. Hence

$$g' = \frac{3}{2}\frac{C'}{M'b'^2} = 0{\cdot}6\frac{\rho_0 + 0{\cdot}037}{\rho_0 + 0{\cdot}052} = 0{\cdot}5973. \quad (6)$$

The uncertainty of the difference of this from 0·6 cannot be put at more than 1 part in 30, or one unit in the fourth place, so long as we suppose the Moon made of one material in a constant state. Increase of temperature towards the centre would make the density more nearly uniform.

Two departures from continuous variation need attention, corresponding to the upper layers and the 480-km. discontinuity in the Earth. It has appeared difficult to explain the latter as due to any new material, and Bernal has attributed it to a change of the crystalline state of olivine at high pressure.* Now the pressure at the centre of the Moon, say $4{\cdot}7 \times 10^{10}$ dynes/cm.^{2}, is reached at a depth of about 140 km. in the Earth. This is far above the discontinuity, and the question of change of state under pressure does not arise.

To estimate the importance of the upper layers, let us suppose a layer of granite of density 2·65 and thickness 40 km. on top of a uniform sphere of radius 1700 km. and density 3·3. Then g' is found to have 0·9915 of its value for a homogeneous sphere, that is, 0·5949. This is probably

* *Observatory*, 59, 268, 1936.

an extreme estimate. The granitic layer in the continents appears to be about 12 km. thick, and is underlain by about 24 km. of some intermediate material with a density of probably about 2·8 or 2·9. Under most of the oceans the granitic layer is probably absent and the intermediate one thinner. On the resonance theory of the origin of the Moon the upper layers in it might be thicker; but the resonance theory now appears unsatisfactory, and if the upper layers in both Earth and Moon separated out from solution in the lower layer we might expect the thicknesses to be about proportional to the radii. Thus the effect of the upper layers may not be more than about $\frac{1}{3}$ of the above estimate, say 0·0017 instead of 0·0051. The effect of compressibility will be present in any case, and is 0·0027. Adding, the total effect of variation of density on g' is 0·0044, giving $g' = 0{\cdot}5956$. The uncertainty of this is mostly due to the thickness of the upper layers; we may take $g' = 0{\cdot}596 \pm 0{\cdot}001$.

Incidentally, this argument from the Moon gives a useful confirmation of our adopted density for the Earth's lower layer, which depended on a somewhat indirect argument.

4. The figure of the Earth affects the motion of the node and perigee through the number $\mathcal{J}$; the figure of the Moon enters through $\mathcal{J}'$ and K'. These three numbers are taken as unknown, but we have a further equation from the rotation of the Moon. The inclination of the Moon's equator to the ecliptic determines β except for a small effect due to f. Thus we have three equations for three unknowns when we use our value of g' to eliminate C'.

For the sidereal motions of the perigee and node per year we have the following (Paper III):—

	$d\varpi$	$d\Omega$
	"	"
Observed	+146435·315 ± 0·006	−69679·398 ± 0·012
Calculated	−	
Principal term	+146426·92 ± 0·05	−69672·04 ± 0·025
Mass of Earth	− 0·68	+ 0·19
Planetary terms	+ 2·54 ± 0·015	− 1·38 ± 0·015
Total	+146428·78	−69673·23
Obs. − Calc.	+ 6·535 ± 0·052	− 6·168 ± 0·031

The observed sidereal motions are based on H. S. Jones's discussion of the longitudes,* the observed motion of the equinox being added. The relativity correction affects the perturbations and the precession equally, and needs to be subtracted to give the remaining Newtonian perturbations. We thus have for the effects of the figures of the Earth and Moon:

$$d\varpi = +6''{\cdot}516 \pm 0{\cdot}052\,; \qquad d\Omega = -6''{\cdot}187 \pm 0{\cdot}031. \tag{1}$$

The theoretical values are:

$$d\varpi = +3891''{\cdot}1\mathcal{J} + 390''\mathcal{J}' - 1027''K'\,; \tag{2}$$

$$d\Omega = -3647''{\cdot}5\mathcal{J} - 470''\mathcal{J}' - 235''K'. \tag{3}$$

* *M.N.*, 85, 36, 1925.

Also de Sitter (Paper I) gives the following equation from the inclination of the Moon's axis :—

$$\beta(1+0{\cdot}0047f)=0{\cdot}0006286\pm 0{\cdot}0000015, \tag{4}$$

which is equivalent to

$$\begin{aligned}1{\cdot}0047\mathcal{J}'+0{\cdot}4976K' &= g'(0{\cdot}0006286\pm 0{\cdot}0000015)\\ &= 0{\cdot}0003746\pm 0{\cdot}0000011.\end{aligned} \tag{5}$$

The uncertainty is composed of 0·0000009 from the inclination and 0·0000006 from g'. Solving (1), (2), (3) and (5), we have

$$\left.\begin{aligned}\mathcal{J} &= 0{\cdot}0016483 \pm 0{\cdot}0000085\\ \mathcal{J}' &= 0{\cdot}000356 \pm 0{\cdot}000024\\ K' &= +0{\cdot}000035 \pm 0{\cdot}000051\end{aligned}\right\}. \tag{6}$$

Most of the uncertainty of $\mathcal{J}'$ and K' can be traced to that of the calculated motion of the perigee in the lunar theory, and that of $\mathcal{J}$ to the motion of the node. These are unlikely to be improved for a considerable time. The inclination of the Moon's axis contributes 0·0000002, 0·000001 and 0·000000 respectively to the uncertainties. The method therefore appears to have reached the limit of its accuracy.

There is a monthly term in the Moon's latitude with coefficient, according to Brown, for 1875, as modified by de Sitter (Paper I),

$$B=-5065\mathcal{J}-0''{\cdot}017=-8''{\cdot}29\pm 0''{\cdot}052, \tag{7}$$

which leads to

$$\mathcal{J}=0{\cdot}0016328\pm 0{\cdot}0000102. \tag{8}$$

De Sitter, however, objects to this determination "on account of the large and uncertain coefficient of T in the observed value." The resulting uncertainty in $\mathcal{J}$, however, is not much more than that of the solution from the perigee and node, and the difference of the two estimates is 0·0000155, which is consistent with the probable errors. Hence we may use this as an additional equation of condition. The results are now :

$$\left.\begin{aligned}\mathcal{J} &= 0{\cdot}0016415\pm 0{\cdot}0000065\\ \mathcal{J}' &= 0{\cdot}0003662\pm 0{\cdot}0000231\\ K' &= 0{\cdot}0000134\pm 0{\cdot}0000466\end{aligned}\right\}. \tag{9}$$

$\mathcal{J}$ agrees almost exactly with de Sitter's revised estimate, but the probable error is 6 times as great. K', as far as this determination goes, is uncertain even as regards its sign. For stability it must be positive. The errors of $\mathcal{J}'$ and K' are opposite in sign, on account of the accuracy of (5). We find

$$K'/2\mathcal{J}'=0{\cdot}018\pm 0{\cdot}069\,;\quad f=0{\cdot}97\pm 0{\cdot}13. \tag{10}$$

The theoretical value of f, on the interpretation that the Moon's ellipticities represent a fossil tide due to rotation and the Earth's tidal action during solidification, is 0·25. $K'/2\mathcal{J}'$ would be 0·6. This can be definitely excluded. I have suggested that a smaller value of K' could originate if the Moon had a large libration in longitude during solidification, but there seem to be

difficulties about this also. It would require the relative rotation to have been almost annulled before solidification ; there could be no such libration while the Moon was fluid, and any contraction during solidification would accelerate the rotation. On the other hand, it is conceivable that if the relative rotation had been annulled a small and rapid contraction would start a large libration, though there are still difficulties about seeing how such a contraction could fail to be spread over several periods of the libration. However this may be, there seems to be an internal inconsistency in the suggestion. The existence of the libration seems to require the Moon to have been solid at the time ; but its use to produce modifications in $K'/2\mathcal{J}'$ depends on a tidal theory and apparently requires a fluid state. I think, therefore, that this suggestion must be abandoned.

The values of f derived from the forced librations in longitude differ seriously. Franz and Stratton, from Schlüter's observations, got f about $0{\cdot}50 \pm 0{\cdot}03$, while Hayn got $0{\cdot}75 \pm 0{\cdot}04$. Hayn, in a rediscussion quoted by de Sitter (Paper I), gets $0{\cdot}85 \pm 0{\cdot}07$, and suggests that "smaller values may be due to an error in the adopted radius vector of Mösting A, which through optical libration can give rise to a spurious oscillation of yearly period, if the observations are made near the time of full Moon." Stratton, however, appears to have discussed this point thoroughly * and found no better agreement by allowing for such an error. As the position of the crater has to be measured from one limb before and the other after full Moon, it seems that errors in the adopted form of the limb may affect the discussion, but so long as both limbs are used equally the error would apparently be thrown into the mean longitude of the crater and not into an annual term. P. Puiseux † gets $(B-A)/C = 0{\cdot}001178$ from measures of photographs ; he gives $19'{\cdot}10$ as the amplitude of the oscillation as seen from the Moon's centre, or $5''{\cdot}2$ as seen from the Earth. This value, however, is impossible. It would make $C < B$, $f = -1$ roughly, and the Moon would be unstable for displacements in the plane of the visible disk.

The general result is, then, that no set of observations agrees with the theoretical value $f = 0{\cdot}25$. The value $0{\cdot}97 \pm 0{\cdot}13$ obtained here would agree with Hayn's later value, and would not be in serious disagreement with his earlier one of 0·75. Otherwise no two sets of data agree. Since f cannot exceed 1·00, the present value is more likely to be too high than too low. It appears worth while to make an alternative solution including Hayn's later value. Then $f = 0{\cdot}87 \pm 0{\cdot}06$; $K'/2\mathcal{J}' = 0{\cdot}07 \pm 0{\cdot}04$. If we definitely adopt $K'/2\mathcal{J}' = 0{\cdot}07$ and 0·11 in turn, we get

$K'/2\mathcal{J}'$	0·07	0·11
$\mathcal{J}$	0·0016453 ± 0·0000058	0·0016464 ± 0·0000058
$\mathcal{J}'$	0·0003487	0·0003362
K'	0·0000488	0·0000740

and finally the solution, with the probable errors, is

* *Mems. R.A.S.*, 59, 257–290, 1909.
† *Ann. de l'Obs. Paris, Mems.*, 32, 65, 1925.

$$\left.\begin{aligned} \mathcal{J} &= 0{\cdot}0016453 \pm 0{\cdot}0000059 \\ \mathcal{J}' &= 0{\cdot}0003487 \pm 0{\cdot}0000125 \\ K' &= 0{\cdot}0000488 \pm 0{\cdot}0000252 \end{aligned}\right\}. \qquad (11)$$

The probable error of $\mathcal{J}$ is only slightly reduced, and the change in $\mathcal{J}$ is within the probable error of either estimate. The additional datum, if it is correct, considerably reduces the uncertainties of $\mathcal{J}'$ and K'. It only makes a last-figure change in $\mathcal{J}' + \frac{1}{2}K'$, the probable error of which is 0·0000011. Finally, comparing the solution with the various data, we have the following comparison :—

	Calc.	Obs.	$O-C$
$d\varpi$	6·488	6·516 ± 0·052	+0·028
$d\Omega$	−6·176	−6·187 ± 0·031	−0·011
$1{\cdot}0047\mathcal{J}' + 0{\cdot}4976K'$	0·0003746	0·0003746 ± 0·0000011	0
B	−8·35	−8·29 ± 0·05	+0·06
f	0·869	0·85 ± 0·07	−0·02

All the residuals are less than the standard errors.

For the Earth we may therefore take

$$\mathcal{J} = 0{\cdot}0016453 \pm 0{\cdot}0000059,$$

as against 0·0016388 given in de Sitter's Paper III and his corrected value 0·0016412. Now when $\mathcal{J}$ is given, e and the main ellipticity term in gravity are determined by the theory of the external field (Paper II) without any hypothesis about the state of stress in the interior. Some second-order terms enter, but these are sufficiently accurately known. An increase in $\mathcal{J}$ makes an equal increase in e, and an equal and opposite one in the coefficient of $\sin^2\phi$ in gravity. Comparing with Paper III we have an increase of 0·0000065 in $\mathcal{J}$; hence the corrected e is

$$0{\cdot}0033675 + 0{\cdot}0000065 = 0{\cdot}0033740 \pm 0{\cdot}0000059\,; \qquad 1/e = 296{\cdot}38 \pm 0{\cdot}51.$$

The corrected coefficient of $g_0 \sin^2\phi$ in gravity is

$$0{\cdot}0052884 - 0{\cdot}0000065 = 0{\cdot}0052819 \pm 0{\cdot}0000059.$$

The variation of gravity from equator to pole is 6 ± 6 milligals less than on de Sitter's formula, and 11 milligals less than that on Heiskanen's formula in a longitude where the term arising from the ellipticity of the equator vanishes. The apparent uncertainty of Heiskanen's value is only 4 milligals, so that the accuracies claimed are comparable; but, as stated above, I think that evaluation of higher harmonics in gravity may lead to some alteration of the coefficient of the second degree one, and the apparent uncertainty may be an underestimate.

De Sitter's calculated value of the lunar parallax, based on $1/e = 296{\cdot}92 \pm 0{\cdot}10$, is $3422''{\cdot}519 \pm 0{\cdot}009 - 0{\cdot}013\delta e^{-1}$, with some other corrections that have not arisen in this paper. Putting $\delta e^{-1} = -0{\cdot}54$, the new probable error being 0·52, we get $3422''{\cdot}526 \pm 0{\cdot}011$. The observed value is affected by the ellipticity (Paper I). To $1/e = 296{\cdot}38$ corresponds the observed value

$3422''{\cdot}40 \pm 0{\cdot}06 \pm 0{\cdot}03 = 3422''{\cdot}40 \pm 0{\cdot}07$. The difference slightly exceeds the standard error of the observed value.

This discussion really concerns only the terms in P_2. There is no reason why the gravitational potential should not contain a term in P_4; but observations of the Moon will not determine it, and its value may differ considerably from that given by the hydrostatic theory. The coefficients of $\sin^2 2\phi$ in the various formulæ mostly rest on the hydrostatic theory, with the exception of the international formula due to Cassinis, who adjusts the P_4 term to make the free surface an exact ellipsoid. It does not matter what we do until we can find the term from observations of gravity. I shall take the form

$$g = g_0(1 + 0{\cdot}005282 \sin^2\phi - 0{\cdot}000007 \sin^2 2\phi). \qquad (12)$$

Then g_0 is calculated gravity on the equator. De Sitter gives $979{\cdot}770 \pm 0{\cdot}002$ gals for gravity at mean latitude ($\sin^2\phi = \frac{1}{3}$). Then $g_0 = 978{\cdot}051$ gals, which is 2 milligals more than Heiskanen's value.

The parameters $\mathcal{J}$, q are defined, ignoring the ellipticity of the equator, by

$$\mathcal{J} = \frac{3}{2}\frac{C-A}{M_1 b^2} = qH = q\frac{C-A}{C}, \qquad (13)$$

where M_1 is the total mass and b the equatorial radius. Thus

$$q = \frac{3}{2}\frac{C}{M_1 b^2} = \frac{\mathcal{J}}{H} = \frac{0{\cdot}0016453 \pm 0{\cdot}0000059}{0{\cdot}0032794 \pm 0{\cdot}0000010} = 0{\cdot}5017 \pm 0{\cdot}0018; \qquad (14)$$

$$\frac{C}{M_1 b^2} = \tfrac{2}{3}q = 0{\cdot}3345 \pm 0{\cdot}0012. \qquad (15)$$

For some purposes it is desirable to have the corresponding quantity for the mean moment of inertia and mean radius. The former is

$$\tfrac{1}{3}C + \tfrac{2}{3}A = C\left(1 - \frac{2}{3}\frac{C-A}{C}\right). \qquad (16)$$

The radius is

$$\begin{aligned} r &= b(1 - 0{\cdot}0033740 \sin^2\phi + 0{\cdot}0000066 \sin^2 2\phi) \\ &= 0{\cdot}9988783b \end{aligned} \qquad (17)$$

in mean latitude. Hence

$$\frac{\frac{1}{3}C + \frac{2}{3}A}{M_1 r^2} = \frac{0{\cdot}3345(1 - \frac{2}{3}H)}{0{\cdot}9989^2} = 0{\cdot}3345. \qquad (18)$$

The correction to mean latitude does not affect $C/M_1 b^2$ to this accuracy.

5. For the Moon, the parameter β is given by (4), and $\beta - \alpha = (1 - f)\beta$; whence

$$\beta = 0{\cdot}0006260 \pm 0{\cdot}0000015; \qquad (1)$$

$$\beta - \alpha = (0{\cdot}13 \pm 0{\cdot}06)\beta = 0{\cdot}0000814 \pm 0{\cdot}0000376. \qquad (2)$$

The result found for f implies that the ellipticity of the Moon's equator is only about a sixth of that of the disk. On the fossil-tide theory the

inequality would be reversed, the ratio being 3 to 1. It appears to follow that when the Moon solidified it was in a state of free rotation. Had it already come to present a constant face to the Earth there seems to be no escape from the latter ratio. In free rotation, on the other hand, tidal force is not available as an explanation of any difference $B-A$ at all, and f should apparently be exactly unity. Then the ellipticity of the equator would not be available to keep one axis permanently towards the Earth, and this would have to be attributed to imperfections of elasticity alone. We cannot attribute it to later tidal action, for if permanent set could take place under the weaker tidal forces at a greater distance, the main ellipticity expressed by $(2C-A-B)/2C$ should have subsided. It seems necessary, therefore, to attribute the main ellipticity to the primitive rotation and $B-A$ to irregular changes arising after solidification and presumably analogous to the ellipticity of the equator of the Earth. Such changes must, however, have occurred long enough ago for the resulting free libration to have disappeared.

An estimate of the rate of rotation at solidification can be made easily. $B-A$ must now be treated as an accidental disturbance, and our datum is

$$\frac{2C-A-B}{2C} = \frac{1+f}{2}\beta = 0{\cdot}94\beta = 0{\cdot}00059. \tag{3}$$

Rotation alone is expressible by a disturbing potential,

$$\tfrac{1}{6}n^2(x^2+y^2-2z^2) = Kr^2S_2, \tag{4}$$

say, where n is the rate of rotation at the time. If the disturbed surface is written

$$r = a(1+eS_2), \tag{5}$$

then as the Moon is nearly homogeneous the condition for a free surface gives

$$\epsilon = \tfrac{5}{2}K\frac{a^3}{fm}, \tag{6}$$

where m is the mass of the Moon.* The ellipticity is then found to be

$$\frac{5}{4}\frac{n^2a^3}{fm} = 0{\cdot}00059, \tag{7}$$

which gives

$$2\pi/n = 3{\cdot}0\times10^5 \text{ sec.} = 3{\cdot}5 \text{ days.}$$

My previous solution gave 6·3 days for the periods of rotation and revolution, assumed equal. The present solution is only an *ad hoc* hypothesis, since it lacks the confirmation from f believed to exist for the previous one ; it is not necessarily an improvement on Laplace's suggestion that the excess ellipticity is due to random changes during solidification.

I take this opportunity of remarking that my hypothesis was anticipated by W. F. Sedgwick † in a paper that appears to have been overlooked.

* *The Earth*, p. 226, 1929.

† *Mess. of Math.*, **27**, 171–173, 1898.

Sedgwick's attention and mine both seem to have been called to the problem by the early values quoted in Routh's *Advanced Rigid Dynamics*, namely,

$$\frac{B-A}{C}=0{\cdot}00564\,; \qquad \frac{C-B}{C}=0{\cdot}00035\,; \qquad \frac{C-A}{C}=0{\cdot}00599,$$

whence $f=0{\cdot}06$. This appears to explain why I thought the observed values consistent with this ratio being 0·25. The later values quoted by Sedgwick and Routh make $f=0{\cdot}5$.

6. Professor F. K. Th. van Iterson has called my attention to a numerical mistake on this subject in *The Earth*, second edition, p. 220. I used the radius of the Earth's core as found from seismology, $C/Mb^2=\frac{1}{3}$, and mean density 5·53, to find the densities of a uniform shell and core for a structure of Wiechert type. The results given should be replaced by

$$\mu=1{\cdot}9197\,; \quad \rho_0=4{\cdot}22\,; \quad \rho_1=12{\cdot}32.$$

The alteration is not serious, and, thanks to Bullen,* we have now a good knowledge of the variation of density within the layers, but if for any purpose uniform layers have to be assumed, these values are better than those in *The Earth*.

7. Professor E. A. Ansel † claims to have found a numerical mistake in Darwin's theory of the figure of the Earth, but it appears to me that Darwin is right. The theory is developed for geocentric latitudes, and the discrepancy arises in the transformation to geographical ones. If we use θ for the geocentric colatitude and ϕ for the geographical latitude, and consider the meridian as an ellipse with major and minor axes a and b, its equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

whence

$$\cot\theta=y/x\,; \qquad \tan\phi=a^2y/b^2x\,; \qquad \cot\theta=(b/a)^2\tan\phi=(1-e)^2\tan\phi\,;$$
$$\cos^2\theta=\sin^2\phi-4e\sin^2\phi\cos^2\phi+O(e^2)\,;$$
$$\sin^2\theta=\cos^2\phi+4e\sin^2\phi\cos^2\phi+O(e^2).$$

Ansel, however, takes $\cot\theta=(1-e)\tan\phi$; thus he gets 2 instead of 4 in these equations. The result is to reverse the sign of the term in $\sin^2 2\phi$ in g.

Ansel calls attention to the fact that Wiechert's theory of the figure of the Earth was a second-order one and preceded Darwin's. It did not use the Radau transformation, and possibly for that reason has been somewhat neglected; but the type of ambiguity in the density distribution that we now recognize as due to Radau's approximation was found.

Summary.—De Sitter's theory of the figure of the Earth is somewhat unsatisfactory because it assumes hydrostatic conditions within the Earth, which do not hold. It has been corrected for the relativity effects, and it is found that the main ellipticity effects at any rate do not show significant

* *M.N., Geophys. Suppl.*, 3, 395–401, 1936.
† *Handbuch der Geophysik*, 1, 668, 1935.

departures from the theory. A rediscussion of the data for the Moon's motion has been carried out, using only the theory of the field outside the Earth. A geophysical analogy shows that the Moon must be nearly homogeneous. The data are found to be consistent with the value $0{\cdot}87 \pm 0{\cdot}06$ for the ratio f in the Moon; the Earth's ellipticity is found to be $1/296{\cdot}38 \pm 0{\cdot}51$, and the corresponding gravity formula is

$$g = 978{\cdot}051\{1 + (0{\cdot}005282 \pm 0{\cdot}000006)\sin^2\phi - 0{\cdot}000007\sin^2 2\phi\}.$$

It is believed that the apparent accuracy of this formula is genuine as far as the term in $\sin^2\phi$ is concerned, and that other formulæ may be in error by larger amounts on account of failure of the methods of analysis to allow for other terms. The ratio C/Mb^2 for the Earth is $0{\cdot}3345 \pm 0{\cdot}0012$. The ellipticities of the Moon have been evaluated. It appears that their high values cannot be explained on the supposition that the Moon solidified when already keeping the same face to the Earth, but they can be explained if it was rotating freely in about 3·5 days.

ON THE VELOCITY ELLIPSOID AND THE GENERAL STAR-STREAMING IN THE REGION AROUND THE SUN.

Bertil Lindblad.

In a recent paper on the theory of stellar systems * the theoretical explanation of the features of the velocity dispersion around the Sun, briefly the "velocity ellipsoid," and its relation to other phenomena of stellar velocity distribution were discussed in considerable detail. It has been found, however, that some of the arguments leading to the adopted theoretical point of view may be developed in a very condensed form, starting from a perfectly general basis, and it is the aim of the present small contribution to present this development, which may perhaps serve to clarify the discussion on this rather complicated subject. As it is not intended to repeat the previous discussion here, we shall have to refer to it as a general background of the present considerations. The author wishes to express his indebtedness to the recent work by Heckmann and Strassl,† which in a general way has been of considerable importance for the present development, although their explanation of the anomalies of the velocity distribution differs in principle considerably from mine.

We consider the distribution of velocities in a region around the Sun which is small compared with the dimensions of the stellar system, and consequently small compared with the dimensions of the orbits of the stars in a fixed co-ordinate system. We assume the field of force to have rotational symmetry, and introduce a fixed co-ordinate system x, y, z with the negative x-axis in the direction of the dynamical centre of the stellar

* *Stockholms Observatoriums Annaler*, 12, No. 4, 1936.

† *Göttingen Veröffentlichungen*, No. 41, 1934; No. 43, 1935.

system and the y-axis in the direction of rotation. We further let the origin $x=y=z=0$ coincide with the projection of the Sun on the central galactic plane at the distance R_0 from the galactic centre. The individual orbits of stars passing through the region considered will be but little curved within the region, and the velocities x', y', z' will be nearly constant. We denote the potential function by V and assume for the potential surface passing through the Sun $V=C$. Essential for the present discussion is a division of the stars into out-going and in-going local orbits. These two groups are of course separated by the sign of the velocity along the radius vector, or with sufficient approximation by the sign of x'. We assume that an orbit intersects the surface $V=C$ in a point x_1, y_1, z_1 with velocity components i_1, i_2, i_3. We consider here the intersection which is nearest to the centre of the region.

For an orbit passing through an arbitrary x, y, z with velocity components x', y', z', the parameters x_1, y_1, z_1 and i_1, i_2, i_3, valid for the intersection of the orbit with the surface $V(x, y, z)=C$, will be determined as functions of x, y, z, x', y', z'. In fact, we have five first integrals of a local orbit of the type $F_i(x, y, z, x', y', z')=c_i$. In the present case the c_i are thus in principle determined by x, y, z, x', y', z'. The parameters $x_1, y_1, z_1, i_1, i_2, i_3$ are then fixed by six equations, namely, the five $F_i(x_1, y_1, z_1, i_1, i_2, i_3)=c_i$ together with $V(x_1, y_1, z_1)=C$. Therefore we may write

$$\left.\begin{aligned} i_1 &= x'[1+f_1(x, y, z, x', y', z')] \\ i_2 &= y'[1+f_2(x, y, z, x', y', z')] \\ i_3 &= z'[1+f_3(x, y, z, x', y', z')] \end{aligned}\right\}. \tag{1}$$

$x'f_1$, $y'f_2$, $z'f_3$ are in general small quantities, and moreover $f_1=f_2=f_3=0$ for $x=x_1$, $y=y_1$, $z=z_1$. The integral of energy may be written

$$I_1=x'^2+y'^2+z'^2-2V=i_1^2+i_2^2+i_3^2-2C. \tag{2}$$

Further, the area integral will be

$$I_2=i_2(R_0+x_1)-i_1y_1.$$

R_0 is in general large compared with x_1 and y_1. We may therefore set with sufficient approximation

$$I_2=R_0i_2. \tag{3}$$

We may remark here that if we had defined i_1, i_2, i_3 in relation to cylindrical co-ordinates R, θ, z, with the linear velocities Π, Θ, Z, thus $i_1=\Pi[1+f_1(R, \theta, z, \Pi, \Theta, Z)]$, etc., the relation (3) would have been $I_2=R_1i_2$, where R_1 differs very little from R_0. For simplicity, however, we prefer to carry out the discussion for the linear co-ordinates x, y, z.

In the actual case we may modify the equation (2) slightly. We may suppose with sufficient approximation that in our neighbourhood

$$V(x, y, z)=V_1(x, y)+V_2(z),$$

which means that we can ignore the dependence of $\frac{\partial V}{\partial z}$ on x and y in our

immediate surroundings. We may set $V_2 = 0$ for $z = 0$. We then define

$$I_1^{(1)} = x'^2 + y'^2 - 2V_1, \qquad I_1^{(2)} = z'^2 - 2V_2.$$

i_1 and i_2 may now be generalised to mean the values of x' and y' for the intersection x_1, y_1, z_1 of the orbit with the cylindrical surface $V_1 = C_1$ passing through the origin, and i_3 to mean the value of z' for the intersection with the plane $V_2 = C_2$. We then have

$$I_1^{(1)} = i_1^2 + i_2^2 - 2C_1, \qquad I_1^{(2)} = i_3^2 - 2C_2. \tag{2a}$$

For C_2 the most appropriate choice will be $C_2 = 0$. Equation (3), of course, remains intact. For cylindrical co-ordinates it should now be exact, as $R_1 = R_0$.

It is clear that i_1, i_2, i_3 may be considered as three first integrals of the motion of a star. We include here formally the case of an imaginary i_1 determined by equations (2*a*) and (3) for orbits which do not intersect $V_1 = C_1$. For the remaining three integrals we may choose

$$s_1 = x - \int_0^t x'\,dt, \qquad s_2 = y - \int_0^t y'\,dt, \qquad s_3 = z - \int_0^t z'\,dt. \tag{4}$$

s_1, s_2, s_3 are thus equal to x, y, z for $t = 0$. We may set

$$s_1 = x - x't + \tfrac{1}{2}x''t^2 - \ldots, \quad s_2 = y - y't + \tfrac{1}{2}y''t^2 - \ldots, \quad s_3 = z - z't + \tfrac{1}{2}z''t^2 - \ldots,$$

and thus approximately, if K_x, K_y, K_z are the components of the central force,

$$s_1 = x - x't + \tfrac{1}{2}K_x t^2, \qquad s_2 = y - y't + \tfrac{1}{2}K_y t^2, \qquad s_3 = z - z't + \tfrac{1}{2}K_z t^2, \tag{5}$$

which we suppose to be valid for a certain interval of time, between t_1 and t_2, say, where $t_1 < 0$, $t_2 > 0$.

We may, of course, define three equivalent integrals in another way, so that t appears explicitly in only one of them. We might choose, for instance, y_1, z_1 and the time t_0 corresponding to the position x_1, y_1, z_1 in the orbit. For the general development of the frequency function, however, the integrals (5) are to be preferred in our present discussion. In a perfectly general frequency distribution, which we may suppose valid, for instance, for the stars of a given mass,

$$f(x, y, z, x', y', z', t)dxdydzdx'dy'dz', \tag{6}$$

the frequency function f will now obey the relation

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z' + \frac{\partial f}{\partial x'}x'' + \frac{\partial f}{\partial y'}y'' + \frac{\partial f}{\partial z'}z'' = 0, \tag{7}$$

and may therefore be written

$$f = F(i_1, i_2, i_3, s_1, s_2, s_3). \tag{8}$$

As a frequency distribution which for small values of t remains *approximately ellipsoidal* we may take, considering for the moment orbits of real i_1,

$$f = Ce^{-\frac{1}{2}Q},$$

where Q is a general development in powers of i_i and s_i, in which, however, the terms of higher order than the second in i_i are assumed to be small. Thus we write

$$Q = \Sigma[D_i i_i + E_{ij} i_i i_j + F_i^{(1)} s_i + F_{ij}^{(2)} s_i i_j + F_{ijk}^{(3)} s_i i_j i_k + \\ + G_{ij}^{(1)} s_i s_j + G_{ijk}^{(2)} s_i s_j i_k + G_{ijkl}^{(3)} s_i s_j i_k i_l + \\ + H_{ijk}^{(1)} s_i s_j s_k + H_{ijkl}^{(2)} s_i s_j s_k i_l + H_{ijklm}^{(3)} s_i s_j s_k i_l i_m + \ldots]. \quad (9)$$

If the distribution is to remain ellipsoidal to the same degree of approximation between t_1 and t_2, we have to consider only the terms which are linear or of the second order in s_i and i_i. We may, however, add intercombination terms containing $\sigma_{ij} i_k$, $\sigma_{ij}\sigma_{kl}$, where

$$\sigma_{ij} = s_i i_j - s_j i_i ; \qquad i \neq j. \quad (10)$$

Assuming $K_x t^2$, $K_y t^2$, $K_z t^2$ to be small, an *approximately steady*, approximately ellipsoidal distribution may be defined in the following way :—

$$Q = \Sigma[D_i i_i + E_{ij} i_i i_j + F_{ij}\sigma_{ij} + G_{ijk}\sigma_{ij} i_k + H_{ijkl}\sigma_{ij}\sigma_{kl}]. \quad (11)$$

This distribution depends on five independent integrals, and is for infinitesimal t and for an infinitesimal domain around origin, in which the orbits may be assumed to be strictly rectilinear, the most general steady ellipsoid.

Among these surfaces there is one of outstanding importance, namely,

$$Q_n = A_1 I_1^{(1)} + A_2 I_1^{(2)} + B_1 I_2^2 + B_2 I_2 = A_1(x'^2 + y'^2) + A_2 z'^2 + \\ + B_1[(R_0 + x)y' - yx']^2 + B_2[(R_0 + x)y' - yx'] - 2A_1 V_1 - 2A_2 V_2. \quad (12)$$

That this expression is a particular case of (11) is evident from the relations (2*a*) and (3) which connect i_1, i_2, i_3 with $I_1^{(1)}$, $I_1^{(2)}$, I_2. We call (12) the "natural" or "normal" velocity ellipsoid. It has immediate connection with the steady ellipsoids studied in the classical papers of Eddington * and Jeans.† We assume here, in general conformity with the rotation theory of the Galaxy, that a velocity distribution in accordance with this ellipsoid forms the dominating feature of the velocity dispersion in our neighbourhood of space, and that the velocity distribution has a general tendency to approach this state. The true significance of this statement may be seen by means of the special method of deriving the velocity ellipsoid for small internal dispersion which has been given by the present writer.‡ We consider here the motion in a rotating co-ordinate system ξ, η, which follows the circular orbit of the same value of I_2 as the particle considered, and assume that the dimensions of the resulting orbits, which are closed curves of elliptical form, are small compared with the dimensions of the stellar system. We may call these orbits the "relative" or "variational" orbits. We then get the normal ellipsoidal velocity distribution, with the vertex line directed towards the centre and the ratio between the axes $\sqrt{1 - \frac{A}{\omega}}$, where A is Oort's constant and ω the angular speed of rotation, if we assume

* *M.N.*, **76**, 37, 1915. † *M.N.*, **76**, 70, 1915.

‡ *Arkiv för Matematik, Astronomi och Fysik*, 20A, No. 17, 1927, Upsala Medd., No. 26 ; *Stockholms Observatoriums Annaler*, **12**, No. 4, 1936, p. 31.

that the mass carried by unit phase-volume in orbits passing through a given point follows a Gaussian distribution

$$f = Ce^{-kc^2}, \tag{13}$$

where c is the small axis of a relative orbit. The essential assumptions made here are that relative orbits of the same size passing through our point do not differ systematically as to mean mass per unit phase-volume, and that the matter is distributed entirely at random along each individual relative orbit when compared with relative orbits close to it. Considering the steady shift of the centres of the "variational" orbits of different I_2 relative to each other in the longitude co-ordinate and the variation of period with size for relative orbits of the same I_2, an approximate realisation of both these assumptions will in general be a natural consequence of the process of mixing due to the orbital motions themselves. The only assumption of cosmological character made is that there is a general concentration of frequency towards the small relative orbits in the system, and that this concentration changes slowly with I_2.

In the more general case (12) the orbits of equal frequency are those which obey the relation $Q_n = \text{const}$. The speed of rotation corresponding to maximum frequency will therefore in the general case differ from the circular velocity at the point in question. A matter of considerable importance is, as shown by Heckmann * and Lindblad,† that no other strictly ellipsoidal steady distribution within a finite region is possible, except for special forms of the field of force, namely, the Newtonian law $K = CR^{-2}$ and the quasi-elastic attraction $K = CR$.

In the course of the general mixing process of the orbital motions the actual velocity distribution at a certain point may, of course, differ from a normal ellipsoid in a number of ways. When we consider the conditions within a limited region of the system, we may now class the disturbances within the region as *secular* variations and *accidental* variations. The secular variations are the same throughout the region for similar orbits, by which we mean here orbits of the same i_1, i_2, i_3. The *accidental* fluctuations are those which depend explicitly on the co-ordinates within the region, whether they occur between similar orbits or along one and the same orbit. They will evidently depend on the parameters s_1, s_2, s_3.

For the secular variations the possibility of a variation with the signs of i_1 and i_3 is of fundamental importance. By means of the signs of i_1 and i_3 we define in- and out-going motions with respect to the centre of the system, as well as up- and down-going motions with respect to the galactic plane. We evidently segregate in this way groups of stars of widely different phases in their orbital motions.

The accidental fluctuations, which depend on the s_i and i_i, will for given i_1, i_2, i_3 occur between motions close in space and in nearly the same orbital phase. It is clear that these fluctuations for a given i_1, i_2, i_3 may

* *M.N.*, **96**, 67, 1935.

† *M.N.*, **96**, 69, 1935. See also *Stockholms Observatoriums Annaler*, **12**, No. 4, 1936, p. 30.

be highly irregular. There is in fact no reason why we should limit the development (9) to second order terms in $s_i i_j$. From the observed fact that the velocity distribution is approximately an ellipsoidal one, there is no reason to assume that a distribution depending explicitly on the s_i stays ellipsoidal to the same degree of approximation in the interval of time between t_1 and t_2. A more reasonable assumption is, no doubt, that the variations due to s_i have in general no systematic character within the small region, and that they may accordingly be assumed to cancel out when we consider the frequency in finite intervals of x, y, z or x', y', z'. We assume the "secular" variations to be so adjusted that in any case the sum of the accidental variations for given small intervals of x', y', z' around a given i_1, i_2, i_3, the sum extended to all points within the region, is zero.

Of course the difference between "secular" and "accidental" variations is not an absolute one. The secular variations will depend on the variation of frequency along the orbits at the epoch $t=0$, when we trace the orbits in a wider region embracing a large part of the stellar system. We may therefore define the secular variations as "large-scale" variations, and the accidental as "small-scale" variations. The stars of different i_1, i_2, i_3 which pass at a certain time through a certain small region converge towards the region from wide distances apart, and thus the infinitely complicated series of events in the past history of the system which determine a variation with the co-ordinates s_i within the region for $t=0$ will change entirely even for small changes in i_1, i_2, i_3. In general we may assume, disregarding real moving clusters, that the stars of the region which at $t=0$ lie in certain intervals dx', dy', dz' around a given i_1, i_2, i_3 are distributed at random in s_i within the part of the region actually covered by the orbits in question. The small-scale variations will thus be uncorrelated for different i_1, i_2, i_3, while this will not necessarily be the case for the large-scale variations.

When taking the moments in x', y', z' for computing the velocity ellipsoid at a certain point x, y, z the accidental deviations may be assumed to cancel out within certain mean errors which *a priori* must be expected to decrease with an increasing number of stars observed. We accordingly neglect the terms containing s_i in (9), and now explain systematic deviations in the velocity ellipsoid, for instance, a deviation of the vertex direction, entirely by the development in i_i. The disturbing terms, *i.e.* the difference between this development and the normal ellipsoid, are of course not confined to powers of the second order in i_i.

We assume i_2 to be positive throughout, in consequence of the general rotational motion of the stars. By means of (2*a*) and (3) we may then write

$$i_1 = \pm \sqrt{I_1^{(1)} - \frac{1}{R_0^2} I_2^2 + 2C_1}, \qquad i_2 = \frac{1}{R_0} I_2, \qquad i_3 = \pm \sqrt{I_1^{(2)} + 2C_2}. \quad (14)$$

The general frequency function in the region may then be written

$$F(i_1, i_2, i_3) = \psi_n(I_1^{(1)}, I_1^{(2)}, I_2), \quad (15)$$

where ψ_n takes the forms ψ_1, ψ_2, ψ_3, ψ_4 in the four cases

$$i_1 > 0, i_3 > 0; \qquad i_1 > 0, i_3 < 0; \qquad i_1 < 0, i_3 > 0; \qquad i_1 < 0, i_3 < 0.$$

We have thus a deformation of the normal ellipsoid (12) of different nature for different signs of i_1 and i_3.

Whereas the observed ellipsoid will be determined by the moments with regard to the distribution (15), the normal ellipsoid may be defined as the approximation of the form (12) to the function

$$\psi = \tfrac{1}{4}(\psi_1 + \psi_2 + \psi_3 + \psi_4), \tag{16}$$

where no regard is paid to the signs of i_1 and i_3. In the course of an undisturbed natural motion in a general field of force of rotational symmetry during a sufficiently long time ψ_n will tend to coincide with ψ. In the general case ψ_n as well as ψ will be secularly variable, as long as they differ at any place, and as long as ψ is not only the local part of a general function which is the same everywhere in the system.

Under certain circumstances the deformation of the normal ellipsoid for motions in a certain direction may be conveniently described as due to a special "stream" in this direction, though we have to admit a certain dispersion in i_i within the "stream." This point of view is supported by observational evidence for some spectral types, especially types A and F, as was first shown by G. Strömberg.* It is further strengthened by the fact that from structural features and dynamical properties of the system we have reasons to expect the appearance of such streams of a special kind. I have shown that a radial density gradient of an amount which may be expected, though still only in a general way, from statistical investigations of stellar distribution at a distance from us between 1000 and 2000 parsecs in the direction of the centre of the stellar system in the Sagittarius region would give rise to certain streamings which are nearly asymptotic to the circular orbits in the region of the gradient. At our point of space the normal velocity ellipsoid will thus be disturbed by out- and in-going streams describing the asymptotic orbits. A slight "secular" shift in I_2 between out- and in-going streams accounts for a vertex deviation. A satisfactory theoretical basis for the treatment of the problem of star-streaming thus appears to be a compromise between Kapteyn's two-stream theory and Schwarzschild's and Charlier's ellipsoidal theory, both seen, however, not merely from a statistical, but from a dynamical point of view.

An indirect confirmation of the existence of the gradient providing the streams is given by the "high-velocity" stars. The marked one-sided distribution of the directions of motion of these stars has been shown to be completely explainable, when we consider the motions of stars of nearly the same I_1 and I_2 as the streams. There seems to be a continuous transition from the star-streams to the showers of "high-velocity" stars. While some groups of high-velocity stars may be said to belong to an extremely wide distribution with low angular speed of rotation, for which the conditions of the nearly circular motions play a minor part, other groups with higher mean speed of rotation no doubt emanate from the dense inner regions, from which they have been accidentally ejected by local variations of the gravitational field in the same way as the streams.

* *Mount Wilson Contr.*, No. 257 ; *Astroph. J.*, 57, 77, 1923.

Another point of great interest in this connection, which has not been explicitly mentioned previously, is furnished by the local clouds of B stars in our neighbourhood as revealed in the work of Charlier * and Shapley.† The break in the "characteristic envelope-curve" of the I_1, I_2 diagram, which gives rise to the instability of the circular motions and to the asymptotic orbits in a certain region of the system, is produced by the density gradient just considered and the general degree of flattening of the interior regions. It is likely that this phenomenon is of fairly late origin in the course of development of the system, which we may suppose to have proceeded from a widely extended gaseous state through condensation into stars and meteoric material into the present less wide but very much flattened form.‡ In the early development of the discontinuity of the envelope a considerable amount of matter is likely to have been expelled from the critical circle into a narrow region of unstable circular orbits. With increasing flattening of the interior regions of the system the region of unstable circular orbits increases in width, and the originally detached matter will be found far out in the outer region of stable circular orbits. The assumption that the clouds of B stars correspond to such detached masses is in the very best accordance with their present motion in nearly circular orbits in a region which is traversed in both directions by the asymptotic orbits of the star-streams, which have nearly the same area constant I_2 as the B stars. Our theory is thus that an originally continuous characteristic envelope-curve has at a certain value of I_2 been gradually broken into the present form, which has two "arrow" points, of which one has just this value of I_2. The star-streams belong to the inner branch, ending at this value of I_2; the B stars, on the other hand, lie close to the outer branch of the present envelope-curve. The conditions are shown by the figure, which illustrates schematically the development of the characteristic envelope-curve, and gives a schematical vertical section through the stellar system.

A gradual extension of the detached masses which proceeds fairly rapidly at the present time gives a ready explanation of a K-effect in the motions of the near B stars. A slight deviation of the axis of rotation of the expelled mass from that of the stellar system as a whole explains further the inclination of the central plane of the distribution of the bright B stars to the galactic plane.

A further confirmation which has already been discussed at length in previous papers is provided by the differential rotation within the regions of the system close to the Sun. The streams alone, taken as secular disturbances in the sense described above, would give a differential rotation W in the radial direction and T in the tangential direction of the form

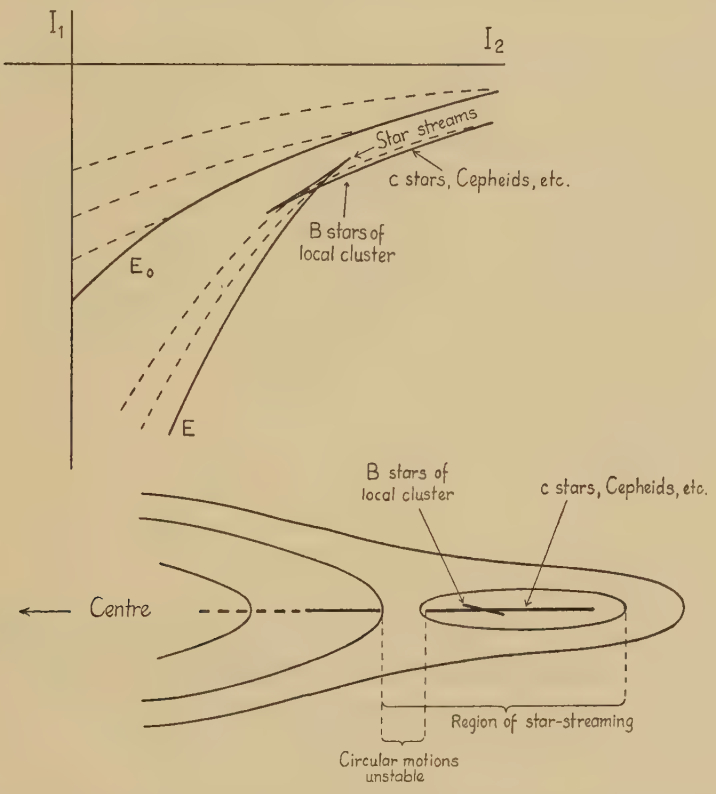
$$\left.\begin{aligned} W &= r[-\omega \tan \phi + A_0 \sin 2(\lambda - \tfrac{1}{2}\phi)] \\ T &= rA_0 \cos 2(\lambda - \tfrac{1}{2}\phi) \\ A_0 &= \omega \sec \phi \end{aligned}\right\} \qquad (17)$$

* *Lund Meddelande*, Ser. II, No. 14, 1916 ; No. 34, 1926.

† *Mount Wilson Contr.*, No. 157 ; *Astroph. J.*, 49, 328, 1918.

‡ Cf. *Stockholms Observatoriums Annaler*, 12, No. 4, 1936, p. 41.

where r is the distance from the Sun, λ the galactic longitude reckoned from the true centre of the system and ϕ the vertex deviation from the true centre. It has been shown that a mixture of the streams with a "normal" ellipsoid gives a good agreement with the results of H. Nordström's * careful discussion of the radial velocities for galactic groups of stars of different spectral



Schematical features of the structure and development of the galactic system. In the I_1, I_2 diagram E_0 is the envelope-curve of a primitive but still mainly gaseous state, E the curve of the present state. The dotted lines represent constant frequency. The corresponding spatial distribution is shown below. The oval curves represent schematically equal space-density for all stars taken together. The heavy lines represent the very massive stars of strong galactic concentration.

type, as well as with the differential effects from the second harmonic terms in the proper motions studied by J. Schilt, J. E. Merrill and S. Holm.

It may be worth while to emphasise again the special importance of the absence of a constant term in T. The streams in our neighbourhood thus tend to define on the average an absolute standard of rest for the motion in galactic longitude, which should lead to an apparent numerical decrease of Oort's constant B, when material belonging to the streams is included.

* *Lund Meddelande*, Ser. II, No. 79, 1936.

Summary

A general expression for the velocity distribution in a region of the stellar system in terms of the integrals of the "local" orbits is given. The special cases of ellipsoidal and stationary distributions are considered. The normal velocity ellipsoid demanded by the rotation theory of the Galaxy is discussed, and the deviations from this ellipsoid are classed as "secular" and "accidental." The secular variations depend on differences between widely different phases in the "relative" orbits, the accidental on variations in the region between motions of nearly the same phase. Neglecting the accidental variations, which are assumed to cancel out when computing the moments of the velocities, the secular variations will appear as disturbing streams through our region. A development of the stream theory starting from general features of the stellar system is given in adherence to earlier work.

The high-velocity stars, the distribution of the groups of B stars in our neighbourhood and the local disturbances in the differential rotation are briefly discussed and shown to be phenomena in general accordance with the theory.

Stockholm Observatory,
Saltsjöbaden:
1936 *September.*

ON THE DISTRIBUTION OF THE ABSORBING ATOMS IN THE REVERSING LAYERS OF STARS AND THE FORMATION OF BLENDED ABSORPTION LINES.

P. Swings and S. Chandrasekhar.

Introduction.—Numerous investigations which have been made regarding the effect of spectral type and surface gravity on the ionization of atoms and on the dissociation of molecules have dealt with the total numbers of atoms or molecules down to a certain optical thickness τ_1. In these calculations the temperature is usually considered as a constant; in most of the cases the pressure is also taken as a constant and equal to half of the pressure at the level corresponding to τ_1.

It is also of some interest to compare carefully the distributions of the absorbing atoms of given elements in the reversing layer of a star, as functions of the depth h or of the optical thickness τ. The problem is the following. Let us consider a star characterized by an effective temperature T_e and surface gravity g, and having an electron pressure p_ϵ at the level corresponding to $\tau = 2/3$; let us suppose that all the atoms are perfectly mixed, with determined abundance ratios. What will be the distributions of specified atoms or ions, in given electronic states, as functions of h or τ? How will these distributions vary in stars of different T_e and g?

This problem may be important from several points of view.* As we shall see later, the distributions depend rather sensitively on the actual values of g and T_e; they might thus give rise to intensities of spectral lines which may appear unusual at first sight; it seems to us that an examination of the distributions of the absorbing atoms may be useful for later discussions on spectral classifications or abundance problems. A crude application of the ionization equation, assuming a uniform temperature and a constant pressure (half the pressure at the bottom), may be insufficient in certain special cases. It may thus happen that, owing to the distributions of T and p_ϵ, the absorbing atoms of an element concentrate preferably in a determined region of the reversing layer and give rise to intensities corresponding to the temperature and pressure of that region.†

On the other hand, Swings and Struve in a recent note ‡ have called attention to the abnormal intensities of the *O*II lines appearing in the wings of H_γ, as well as of the Ca^+ line superposed on a wing of H_ϵ. Their general result shows a marked departure from the application of Eddington's formula, and it seems tempting to attribute the observed effect to some kind of stratification. Owing to the great microphotometric difficulties involved in this problem, it would be important to rediscuss the observations of Swings and Struve; this would bring complementary observational information which would be valuable in connection with the distribution of absorbing atoms. In their paper, Swings and Struve suggest some kind of chemical stratification of the elements. But actually a similar result may be obtained if all the atoms are perfectly mixed and if, owing to the effects of T and p_ϵ, the distributions of the absorbing elements are different.

It is obvious that this question is also of some importance for the molecular problem in late-type stars. This will not be considered in this note, which deals only with the case of atoms. Actually, the examples which will be considered here are those which correspond to the observations, namely:

(*a*) In the early A-type stars: distributions of the H and Ca^+ atoms giving rise respectively to the Balmer and to the H and K lines.

(*b*) In the early B-type stars: distributions of the H and O^+ atoms absorbing respectively the Balmer and the visible *O*II lines.

It would be interesting to investigate theoretically (following a method similar to that of this paper) the distribution of the excitation and ionization of the different abundant elements in the atmosphere of ζ Aurigæ, for which direct observations are now available. §

* In a recent paper (*Ap. J.*, **83**, 202, 1936) R. Wildt has reinvestigated the equilibrium of stellar atmospheres under a temperature gradient; if the temperature gradient maintained by the radiation flux is smaller than the adiabatic gradient of the stellar matter, the different constituents of the atmosphere will be separated by diffusion; this gives thus a chemical stratification, which is quite different from the physical distribution examined here.

† In connection with this, see the recent discussions on the spectral classification of early-type stars: Struve, *Ap. J.*, **78**, 73, 1933; Russell, Payne-Gaposchkin and Menzel, *ibid.*, **81**, 107, 1935; E. G. Williams, *ibid.*, **83**, 305, 1936.

‡ *Ap. J.*, **83**, 238, 1936.

§ W. H. Christie and O. C. Wilson, *Ap. J.*, **81**, 426, 1935.

1. *General Formulæ.*—Following the notation of Fowler and Milne, let

x_0 be the fraction of any atomic element in the neutral state and in the lowest electronic level ;
x_1 the fraction of singly ionized atoms ;
x_2 the fraction of doubly ionized atoms ;
$(n_r)_s$ the fraction of the atoms of one type which are r times ionized and in their sth state ;
χ_r the ionization potential of the r times ionized atom in its lowest level ;
$(\chi_r)_s$ the ionization potential in the sth level ;
$(\varpi_r)_s$ the statistical weight of that sth level ;
$u_r(T)$ the partition factors.

We have the following relations :—

$$\frac{x_{r+1}}{x_r} = \frac{0{\cdot}664}{p_\epsilon} \cdot \frac{u_{r+1}(T)}{u_r(T)} \cdot T^{5/2} \cdot e^{-\chi_r/kT} ; \tag{1}$$

$$\Sigma x_r = 1 ; \tag{2}$$

$$(n_r)_s = \frac{x_r \cdot (\varpi_r)_s \cdot e^{-\frac{\chi_r - (\chi_r)_s}{kT}}}{u_r(T)} . \tag{3}$$

(a) *Hydrogen Atoms.*—For neutral atoms in the sth state we have

$$(n_0)_s = \frac{(\varpi_0)_s \cdot e^{(\chi_0)_s/kT}}{u_0(T) \cdot e^{\chi_0/kT} + 0{\cdot}664 \cdot u_1(T) \cdot \dfrac{T^{5/2}}{p_\epsilon}} . \tag{4}$$

In this formula we must introduce the values corresponding to the hydrogen atom : *

$$\chi_0 = 13{\cdot}54 ; \quad (\chi_0)_s = 3{\cdot}385 ; \quad u_0(T) = 2 ; \quad u_1(T) = 1 ; \quad (\varpi_0)_1 = 8.$$

(b) *Singly Ionized Oxygen Atoms.*—At the temperatures of the early B-type stars there is almost no neutral oxygen left, as it appears from an immediate application of the ionization formula (1).† Consequently, the formula for *O*II is exactly similar to that for hydrogen :

$$(n_1)_s = \frac{(\varpi_1)_s \cdot e^{(\chi_1)_s/kT}}{u_1(T) \cdot e^{\chi_1/kT} + 0{\cdot}664 u_2(T) \cdot \dfrac{T^{5/2}}{p_\epsilon}} . \tag{5}$$

The specific factors are : ‡

$$\chi_1 = 35 ; \quad (\chi_1)_s = 11{\cdot}53 ;$$
$$u_1(T) = 4 + 10e^{-3{\cdot}37/kT} + 6e^{-5{\cdot}04/kT} ;$$
$$u_2(T) = 9 ;$$
$$(\varpi_1)_s = 6.$$

* The ionization potentials will be expressed in electron-volts.

† For the range of electron pressures which seem admissible and for $T = 20000°$, x_1/x_0 varies from 10^4 to 10^8.

‡ We have taken the same values as were adopted by R. H. Fowler and which concern $\lambda\lambda$ 4417 and 4415 ; for any other visible line the calculations are very similar.

(c) *Singly Ionized Calcium Atoms.*—For the normal lines of these ionized atoms we must retain three stages ; in other words, we must apply formula (1) for $r=0$ and $r=1$, and consider formula (3) for $r=1$ and the ground level. We get in that way :

$$n_1 = \frac{\varpi_1}{u_1(T)+u_0(T)\cdot\frac{p_\epsilon}{0\cdot664}\cdot T^{-5/2}\cdot e^{\chi_0/kT}+u_2(T)\cdot\frac{0\cdot664}{p_\epsilon}\cdot T^{5/2}\cdot e^{-\chi_1/kT}}\,; \qquad (6)$$

with the following values of the factors :

$$\chi_0=6\cdot08\,; \quad \chi_1=11\cdot82\,; \quad u_0(T)=1\,; \quad u_1(T)=2+10e^{-\frac{1\cdot69}{kT}}\,; \quad u_2(T)=1.$$

2. *Variation of T and p_ϵ in the Stellar Atmosphere.*—In the early-type stars which are considered here we may assume that, inside of a reversing layer, the electron pressure p_ϵ is a constant fraction of the gas pressure. We know that the temperature is related to the optical thickness τ by the Milne formula

$$T^4=\frac{T_e^4}{2}(1+\tfrac{3}{2}\tau). \qquad (7)$$

For the distribution of pressures we shall use the following formula :—*

$$p_\epsilon=\alpha T^{19/4}\cdot\sqrt{1-\left(\frac{T_0}{T}\right)^{19/2}}, \qquad (8)$$

α being a constant, and T_0 the temperature at the boundary ($T_0=0\cdot871\,T_e$). All the calculations will be made for three values of α, namely :

α_0, which will correspond to an intermediary star ;
$100\alpha_0$, corresponding to a dwarf ;
$\alpha_0/100$, corresponding to a giant.

3. *Hydrogen and Ca^+ Atoms in the A0 Stars.*—We shall assume here :

$$T_e=10000^\circ,$$

and

$$\alpha_0=3\cdot10^{-17}.$$

This value of α_0 gives an electron pressure 270 dynes/cm.$^{-2}$ for $\tau=2/3$, which seems quite normal.

Table I gives the results of the calculations for the three values of the constant α. The maximum value of τ which has been considered is $\tau=4$, which, following Milne, defines the greatest depth to which we can see in the photosphere at the centre of the solar disk. Evidently, the most interesting part of Table I concerns the values of τ extending from $\tau=\cdot01$ to $\cdot666$.

Table I brings the following results :—

(a) *Case $\alpha=\alpha_0$.*—The fraction of absorbing H atoms increases steadily with increasing τ, varying from $3\cdot24\,.\,10^{-6}$ at $\tau=\cdot01$ to $25\cdot7\,.\,10^{-6}$ at $\tau=2/3$. On the other hand, the fraction of Ca^+ absorbing atoms increases slowly, reaches a maximum at about $\tau=\cdot05$ or $\cdot10$, and then decreases slowly. This

* S. Chandrasekhar, *M.N.*, **92**, 186, 1932.

TABLE I

Distribution of the Absorbing H^- and Ca^+ Atoms in A0-type Stars

τ	T	$\frac{\alpha}{\alpha_0}p_\epsilon$	With $\alpha=\alpha_0$			With $\alpha=100\alpha_0$			With $\alpha=\alpha_0/100$		
			nBal. in Units 10^{-6}	$n(Ca^+)$ in Units 10^{-2}	$\frac{n\text{Bal.}}{n(Ca^+)}$ in Units 10^{-5}	nBal. in Units 10^{-6}	$n(Ca^+)$ in Units 10^{-1}	$\frac{n\text{Bal.}}{n(Ca^+)}$ in Units 10^{-5}	nBal. in Units 10^{-7}	$n(Ca^+)$ in Units 10^{-3}	$\frac{n\text{Bal.}}{n(Ca^+)}$ in Units 10^{-5}
0·01	8440	25·71	3·24	44·8	0·719	3·46	6·62	0·523	4·34	13·24	3·277
·02	8470	34·77	3·45	47·9	·720	3·65	6·60	·553	5·54	16·52	3·354
·05	8560	57·21	4·05	51·5	·786	4·22	6·56	·643	8·07	22·22	3·632
·1	8710	82·32	5·11	51·5	·992	5·34	6·48	·824	10·30	23·26	4·43
·2	8980	122·97	7·65	48·9	1·56	8·02	6·28	1·27	12·92	20·10	6·43
·5	9670	219·39	18·3	38·1	4·80	20·54	5·92	3·47	15·42	10·26	15·03
·666	10000	269·4	25·7	32·5	7·91	30·53	5·74	5·32	15·62	7·22	21·6
1·0	10575	368·52	42·8	23·8	17·98	58·53	5·46	10·72	15·39	4·12	37·3
1·5	11290	522·48	66·1	7·7	85·8	116·7	5·14	22·7	14·89	2·16	68·9
2	11890	669·84	83·1	5·3	157	196·04	4·88	40·2	14·13	1·34	105
3	12870	985·86	102	2·96	345	411·3	4·44	92·7	13·31	0·68	196
4	13680	1322·64	110	1·88	585	688·4	4·08	169	12·83	0·40	321

means that, on the average and in comparison with the *H* and *K* lines, the hydrogen lines will be formed in deeper layers.

(b) *Case* $\alpha=100\alpha_0$ (*Dwarf*).—A similar result is reached. The increase of the hydrogen-absorbing fraction is even faster than in the first case, while the Ca^+ fraction is decreasing continuously and slowly.

(c) *Case* $\alpha=\alpha_0/100$ (*Giant*).—The increase of the hydrogen fraction is much slower than for $\alpha=\alpha_0$. The Ca^+ fraction reaches a maximum at $\tau=0{\cdot}10$; from there it decreases fast, and is at $\tau=2/3$ only a third of its maximum value. In all cases the ratio $\frac{n(\text{Balmer})}{n(Ca^+)}$ is increasing steadily with τ, reaching higher and higher values in the deeper layers of the atmosphere.*

4. *Hydrogen and Ionized Oxygen in Early B-type Stars.*—We shall assume here

$$T_e=20{,}000^\circ \quad \text{and} \quad \alpha_0=2\cdot10^{-9}.$$

This value of α_0 gives $p_\epsilon=50$ dynes/cm.$^{-2}$ at $\tau=2/3$. Table II gives the results of the calculations. It shows that the behaviour of the hydrogen-absorbing atoms is the same for giant, dwarf and intermediary stars, but that it is extremely different for the O^+ atoms. In the giant ($\alpha=\alpha_0/100$),

* As we assume that p_ϵ is proportional to the gas density, it would be an easy matter to convert these fractions n into actual numbers of atoms; this may be useful for a discussion of certain spectra, but not for the comparison between two atoms.

TABLE II

Distribution of the Absorbing H and O^+ Atoms in Early B-type Stars

τ	T	$\frac{\alpha}{\alpha_0 p_e}$	With $\alpha=\alpha_0$			With $\alpha=100\alpha_0$			With $\alpha=\alpha_0/100$		
			nBal. in Units 10^{-8}	$n(O^+)$ in Units 10^{-8}	$\frac{n\text{Bal.}}{n(O^+)}$ in Units 10^{-1}	nBal. in Units 10^{-6}	$n(O^+)$ in Units 10^{-7}	$\frac{n\text{Bal.}}{n(O^+)}$ in Units 10	nBal. in Units 10^{-10}	$n(O^+)$ in Units 10^{-9}	$\frac{n\text{Bal.}}{n(O^+)}$ in Units 10^{-1}
0·01	16880	4·617	1·536	8·397	1·829		1·115	1·38		3·283	0·469
·02	16940	6·229	2·041	9·258	2·204	*Idem*	1·17	1·74	*Idem*	4·225	·484
·05	17120	10·263	3·197	11·505	2·78		1·395	2·28		6·216	·514
·1	17420	14·771	4·234	14·72	2·88	$\alpha=\alpha_0$	1·806	2·34	$\alpha=\alpha_0$	7·534	·56
·2	17960	22·044	5·479	21·39	2·56		2·832	1·93		8·388	·65
·5	19340	39·328	6·941	38·65	1·79		7·926	0·87		7·395	·94
·666	20000	48·31	7·326	43·84	1·67		12·27	·59		6·719	1·09
1	21150	66·087	7·818	45·92	1·70		24·4	·32		5·593	1·22
1·5	22580	93·7	8·384	41·82	2·00		49·02	·17		4·524	1·88
2	23780	120·127	8·657	36·34	2·38		77·68	·11		3·776	2·29
3	25740	176·809	9·192	29·85	3·08		126·5	·073		2·961	3·11
4	27360	237·191	9·683	25·15	3·85		154·5	·062		2·532	3·82

$n(O^+)$ remains almost constant from $\tau=\cdot01$ until $\tau=4$; while in the dwarf ($\alpha=100\alpha_0$) it increases from $1\cdot115 \cdot 10^{-7}$ to $154\cdot5 \cdot 10^{-7}$. In the dwarf it appears from the behaviour of $\frac{n(\text{Balmer})}{n(O^+)}$ that, when compared with the Balmer lines, the O^+ lines have a tendency to originate rather from deeper layers * (see the variation of $n(\text{Bal})/n(O^+)$ between $\tau=0\cdot1$ and $\tau=0\cdot666$).

5. *The Formation of Blended Absorption Lines in a Stratified Atmosphere.*— Before proceeding to a detailed theoretical discussion of the effect observed by Swings and Struve, it would be necessary to get more observations and more accurate measures. Nevertheless, owing to the general interest of the problem and its various possible applications, it seems worth while to examine what the effect of the different distributions of the absorbing atoms will be on the intensities of blended lines. It is quite certain that the general result of Swings and Struve indicating a marked departure from a straightforward application of Eddington's formula is essentially correct, and the bearing of a plausible "stratified atmosphere" on the formation of superposed absorption lines is well worth examining.

We consider the simplified model of a plane atmosphere containing the absorbing element 1 in the outer region from $\tau=0$ to $\tau=\tau_0$ (say), and the absorbing element 2 at optical depths greater than τ_0. (τ is the optical depth

* It must be remembered that photometric observations of the type carried on by Swings and Struve are only possible when the Balmer lines have rather broad wings, *i.e.* in stars having a dwarf character.

measured in the continuous spectrum in the immediate background of the line.)

Let

κ_ν be the coefficient of continuous absorption;

l_1 and l_2 the coefficients of scattering or of line absorption due to the elements 1 and 2 respectively; *

η_1 and η_2 the customary ratios l_1/κ_ν and l_2/κ_ν respectively.

Further, let

$$\mathfrak{J}_\nu = \tfrac{1}{2}\int_0^\pi I_\nu \sin\theta d\theta; \qquad F_\nu = 2\int_0^\pi I_\nu \sin\theta\cos\theta d\theta; \qquad K_\nu = \tfrac{1}{2}\int_0^\pi I_\nu \sin\theta\cos^2\theta d\theta.$$

Let B_ν be the Planck function.

The equation of transfer is as usual:

$$\cos\theta\frac{dI_\nu}{\rho dx} = -(\kappa_\nu + l_1 + l_2)I_\nu + (l_1+l_2)\mathfrak{J}_\nu + \kappa_\nu B_\nu.$$

Multiplying the equation of transfer successively by $\frac{1}{2}\sin\theta d\theta$ and $\frac{1}{2}\sin\theta\cos\theta d\theta$ and integrating from O to π, we get

$$\frac{1}{4}\frac{dF_\nu}{d\tau} = \mathfrak{J}_\nu - B_\nu, \tag{9}$$

and

$$\frac{dK_\nu}{d\tau} = \tfrac{1}{4}(1+\eta_i)F_\nu, \qquad \begin{cases} i=1 \text{ for } \tau \leqslant \tau_0 \\ i=2 \text{ for } \tau > \tau_0 \end{cases} \tag{10}$$

where we have introduced the optical depth τ by the equation

$$d\tau = -\kappa_\nu \rho dx.$$

We consider the case where η_1 and η_2 are constants. Equation (10) then leads to

$$\frac{d^2K_\nu}{d\tau^2} = (1+\eta_i)(\mathfrak{J}_\nu - B_\nu);$$

or with the usual approximation $\mathfrak{J}_\nu = 3K_\nu$,

$$\frac{d^2\mathfrak{J}_\nu}{d\tau^2} = 3(1+\eta_i)(\mathfrak{J}_\nu - B_\nu). \tag{11}$$

We shall now consider what Eddington calls the "standard case," where we have with sufficient accuracy

$$B_\nu = B_0(1+\tfrac{3}{2}\tau) = \tfrac{1}{2}F(0)(1+\tfrac{3}{2}\tau). \tag{12}$$

In (12), $F(0)$ is the emergent flux in the continuous spectrum in the immediate background of the lines. Equation of the Milne type (12) is strictly true for the integrated radiation for grey material in radiative equilibrium,

* l_1 and l_2 are, of course, functions of the frequency ν, but we have suppressed the suffixes.

but in certain special cases it will also be true in the separate frequencies.* We shall accordingly consider this case first. A better approximation is considered in § 9.

Combining equations (11) and (12), we have

$$\frac{d^2(\mathcal{J}_\nu - B_\nu)}{d\tau^2} = 3(1+\eta_i)(\mathcal{J}_\nu - B_\nu). \tag{13}$$

The required solution of (13) is

$$\mathcal{J}_\nu = B_0(1+\tfrac{3}{2}\tau) + \alpha e^{-q_1\tau} + \beta e^{+q_1\tau}, \qquad (\tau < \tau_0), \tag{14'}$$

$$\mathcal{J}_\nu = B_0(1+\tfrac{3}{2}\tau) + \gamma e^{-q_2\tau}, \qquad (\tau \geqslant \tau_0), \tag{14''}$$

where

$$q_1^2 = 3(1+\eta_1); \qquad q_2^2 = 3(1+\eta_2).$$

In (14), α, β and γ are constants of integration.

By differentiation of (14) we find

$$\tfrac{3}{4}(1+\eta_1)F_\nu = \tfrac{3}{2}B_0 + q_1(\beta e^{q_1\tau} - \alpha e^{-q_1\tau}), \qquad (\tau < \tau_0) \tag{15'}$$

$$\tfrac{3}{4}(1+\eta_2)F_\nu = \tfrac{3}{2}B_0 - q_2\gamma e^{-q_2\tau}. \qquad (\tau \geqslant \tau_0) \tag{15''}$$

The boundary conditions are, that at $\tau = \tau_0$ the $\mathcal{J}$'s given by (14′) and (14″) should be the same, and similarly the F's given by (15′) and (15″) should also be the same. Finally, at $\tau = 0$ we use the boundary condition that $F = 2\mathcal{J}$. These conditions yield the following equations for the determination of the constants α, β and γ :—

$$\left.\begin{aligned} &\alpha(1+\eta_1+\tfrac{2}{3}q_1) + \beta(1+\eta_1-\tfrac{2}{3}q_1) + \eta_1 B_0 = 0, \\ &\alpha e^{-q_1\tau_0} + \beta e^{+q_1\tau_0} - \gamma e^{-q_2\tau_0} = 0, \\ &\alpha e^{-q_1\tau_0}(1+\eta_2)q_1 - \beta e^{+q_1\tau_0}(1+\eta_2)q_1 - \gamma e^{-q_2\tau_0}(1+\eta_1)q_2 + \tfrac{3}{2}B_0(\eta_1-\eta_2) = 0. \end{aligned}\right\} \tag{16}$$

6. *Formulæ for the Residual Intensity.*—For this problem we need only the two constants α and β to determine the emergent flux at $\tau = 0$. Using the abbreviations

$$\xi_1 = (1+\eta_1) \quad \text{and} \quad \xi_2 = (1+\eta_2),$$

we obtain finally that

$$\tfrac{1}{2}\xi_1 F_\nu(0) = B_0 + q_1 B_0 \frac{2\xi_1(\xi_1-\xi_2) + \tfrac{2}{3}(\xi_1-1)[(\xi_2 q_1 + \xi_1 q_2)e^{q_1\tau_0} - (\xi_2 q_1 - \xi_1 q_2)e^{-q_1\tau_0}]}{(\xi_2 q_1 - \xi_1 q_2)(\xi_1 - \tfrac{2}{3}q_1)e^{-q_1\tau_0} + (\xi_2 q_1 + \xi_1 q_2)(\xi_1 + \tfrac{2}{3}q_1)e^{q_1\tau_0}}.$$

The residual intensity is obtained by dividing $F_\nu(0)$ by $2B_0$ (*cf.* equation (12)), and we have

$$r_\nu(\xi_1;\ \xi_2) = \frac{1}{\xi_1}\left\{1 + q_1 \frac{2\xi_1(\xi_1-\xi_2) + \tfrac{2}{3}(\xi_1-1)[(\xi_2 q_1 + \xi_1 q_2)e^{q_1\tau_0} - (\xi_2 q_1 - \xi_1 q_2)e^{-q_1\tau_0}]}{(\xi_2 q_1 - \xi_1 q_2)(\xi_1 - \tfrac{2}{3}q_1)e^{-q_1\tau_0} + (\xi_2 q_1 + \xi_1 q_2)(\xi_1 + \tfrac{2}{3}q_1)e^{q_1\tau_0}}\right\}. \tag{17}$$

One easily verifies from (17) that for either of the cases (i) $\tau_0 = 0$, (ii) $\tau_0 \to \infty$ or (iii) $\xi_1 = \xi_2$ it reduces to the "classical formula"

$$r_\nu = \frac{1+\tfrac{2}{3}q}{1+\eta+\tfrac{2}{3}q}, \qquad \begin{pmatrix} \tau_0 = 0 \\ \tau_0 \to \infty \\ \xi_1 = \xi_2 \end{pmatrix}. \tag{17'}$$

* See A. S. Eddington, *M.N.*, **89**, 620, 1929, § 5 of this paper.

For an absorption line due to element 1 only we have to take $\xi_1 > 1$ and $\xi_2 = 1$, and similarly for an absorption line due to element 2, $\xi_1 = 1$ and $\xi_2 > 1$. But if the *absorption line of element* 1 *appears in the wing of the absorption line due to element* 2, then the residual intensity R_ν in the line due to element 1 will be given by

$$R_\nu(1 \text{ in } 2) = \frac{r_\nu(\xi_1(\nu)\,;\ \xi_2(\nu))}{r_\nu(\xi_1(\nu)\,;\ 1)}, \tag{18}$$

where r_ν is given by (17). Similarly,

$$R_\nu(2 \text{ in } 1) = \frac{r_\nu(\xi_1(\nu)\,;\ \xi_2(\nu))}{r_\nu(1\,;\ \xi_2(\nu))}. \tag{18'}$$

7. *The Coefficients* ξ_1 *and* ξ_2.

(a) *Determination of* ξ_1.—We consider an absorption line due to element 1 at frequencies at which there is no blending with an absorption line due to element 2. Then $\xi_2 = 1$ (or $\eta_2 = 0$). Further, $q_2 = \sqrt{3}$. By (17), then,

$$r_\nu(\xi_1\,;\ 1) = \frac{1}{\xi_1}\left\{1 + 2q_1(\xi_1 - 1)\frac{\xi_1 + \frac{1}{3}[(\sqrt{3}\xi_1 + q_1)e^{q_1\tau_0} + (\sqrt{3}\xi_1 - q_1)e^{-q_0\tau_1}]}{(\sqrt{3}\xi_1 + q_1)(\frac{2}{3}q_1 + \xi_1)e^{q_1\tau_0} - (\sqrt{3}\xi_1 - q_1)(\xi_1 - \frac{2}{3}q_1)e^{-q_1\tau_0}}\right\}. \tag{19}$$

In order to facilitate subsequent work with the above formula, we have tabulated in Table III a number of values of ξ_1 and the corresponding calculated central absorptions * for three values of τ_0.

TABLE III

Relation between ξ_1 *and* $r_\nu(\xi_1\,;\ 1)$

ξ_1	Percentage Central Absorption for $\tau_0 = \frac{1}{6}$	Percentage Central Absorption for $\tau_0 = \frac{1}{3}$	Percentage Central Absorption for $\tau_0 = \frac{1}{2}$
1·1	1·2	2	2·6
1·2	2·3	3·9	5·2
1·3	3·4	5·7	7·4
1·8	8·5	13·8	17·2
2	10·4	16·6	20·5
2·5	14·8	22·8	27·4
3	18·7	28	33
4	25·2	36·3	41·5
5	31·2	42·8	47·8
10	49·6	60·4	63·8
15	59·8	68·5	70·9

* In Table III and also in Table IV we have used the percentages of central absorption instead of the residual intensities.

(b) *Determination of* ξ_2.—We next consider an absorption line due to element 2 at frequencies at which there is no blending with an absorption line due to element 1. Then $\xi_1 = 1$ and $q_1 = \sqrt{3}$. Then

$$r_\nu(1\ ;\ \xi_2) = 1 - \frac{\xi_2 - 1}{\xi_2(1{\cdot}0774e^{1{\cdot}732\tau_0} - 0{\cdot}0774e^{-1{\cdot}732\tau_0}) + \sqrt{\xi_2}(1{\cdot}0774e^{1{\cdot}732\tau_0} + 0{\cdot}0774e^{-1{\cdot}732\tau_0})}. \quad (20)$$

Table IV gives calculated percentage absorption as a function of ξ_2 and τ_0.

TABLE IV

Relation between ξ_2 *and* $r_\nu(\xi_2\ ;\ 1)$

ξ_2	Percentage Central Absorption for $\tau_0 = \frac{1}{6}$	Percentage Central Absorption for $\tau_0 = \frac{1}{3}$	Percentage Central Absorption for $\tau_0 = \frac{1}{2}$	ξ_2	Percentage Central Absorption for $\tau_0 = \frac{1}{6}$	Percentage Central Absorption for $\tau_0 = \frac{1}{3}$	Percentage Central Absorption for $\tau_0 = \frac{1}{2}$
1·1	3·2	2·4	1·8	15	53·2	39·2	29·4
1·2	6·1	4·5	3·4	20	55·3		
1·3	8·6	6·4	4·8	30	58·5		
1·5	12·9	9·6	7·1	40	60·5		
1·8	17·8	13·3	9·9	50	61·6		
2	20·5	15·3	11·5	60	62·7		
2·5	25·8	19·2	14·4	70	63·2		
3	29·8	22·2	16·4	80	63·8		
4	35·1	26·1	19·5	200	87·4		
5	39·2	28·8	21·6	1000	69·9		
10	48·6	36	27	10000	71·7		

8. *Application to the Intensities of Superposed Lines.*—Naturally, a direct application of formula (17) constitutes a very crude simplification of the real physical conditions. We have, however, seen in the first part of this paper that the absorbing elements have different distributions, but a sharp stratification of the kind assumed in formula (17) is evidently not present.* Nevertheless, we shall examine if the application of (17) to the observations of Swings and Struve leads to better agreement than the application of Eddington's formula.

(a) *The H Line of* Ca^+ *Appearing in the Wing of the Hydrogen Line* H_ϵ *(Early A-type Stars).*—Here Ca^+ is the "element 1" and hydrogen the "element 2" of §§ 6 and 7—in agreement with Table I. Application of (17), assuming $\tau_0 = 1/6$, leads to the results indicated in Table V.

In this table the second column gives the percentages of absorption which the *H* line of Ca^+ would exhibit if it were outside the wing of H_ϵ. This was done by using the measured absorption in the *K* line and applying the square-root law for the absorptions. The third column gives the

* The crude aspect of this assumption appears numerically when we want to determine the value of ξ for very strong central absorptions.

measured percentages of absorption for the wing H_ϵ—in the immediate background (in H_ϵ) of the H line of Ca^+. The fourth and fifth columns give the corresponding values of ξ_1 and ξ_2 obtained by interpolation among the values given in Tables III and IV for $\tau_0 = 1/6$. In column 6 we give the residual intensities, starting from the wing of H_ϵ and calculated according to formula (17)—*i.e.* calculating $r(\xi_1\,;\ \xi_2)$ for the values ξ_1 and ξ_2 given in columns 4 and 5. Column 7 gives the factors by which the absorptions indicated in column 2 have to be *reduced* in order to obtain the values given in column 6. The figures in column 8 are the observed reduction factors. Finally, the last column gives the reduction factors, which are obtained without assuming a stratification and by a straightforward application of Eddington's formula.

TABLE V *

Intensities of the H line of Ca^+

(1) Star	(2) Absorption 1	(3) Absorption 2	(4) ξ_1	(5) ξ_2	(6) Calculated Intensity of H	(7) Calculated Reduction Factor	(8) Observed Reduction Factor	(9) Reduction Factor Calculated Classically
α Cygni	52	25	11	2·4	45	1·15	1·02	1·28
η Leonis	30	36	4·8	4·1	22·4	1·34	1·22	1·86
α Can. Maj.	34	62	6	55	16	2·12	1·84	4·26

We see that a stratified atmosphere with $\tau_0 = 1/6$ brings out a very good agreement between the theory and the observations. When considering Table I the value for $\tau_0 = 1/6$ does not seem unlikely ; the observed facts would thus indicate that the difference of distribution between the Ca^+ and hydrogen atoms in the atmospheres of the early A-type stars is perhaps roughly equivalent to a sharp separation at about $\tau_0 = 1/6$.

Any further discussion would be premature at this stage of the measures.

(b) *OII Lines Appearing in the Wings of Hγ.*—Now hydrogen would correspond to "element 1" and ionized oxygen to "element 2"—in agreement with Table II. The calculation has been made only for the *O*II line 4345·57 of θ Ophiuchi and for $\tau_0 = 1/6$. Application of (17) leads to a calculated central absorption of 2·2 per cent. as against the observed figure of 2 per cent., thus explaining one of the largest discrepancies observed by Swings and Struve.

9. *The Formation of Blended Absorption Lines in a Stratified Atmosphere (a more General Treatment).*—So far we have restricted ourselves to the so-called "standard case," assuming for B_ν the expression (12). This is sufficient, as is well known, for most purposes, but in view of the agreement between the "theory" and observations obtained in § 8, it is necessary to

* The star α Lyræ, for which measures had also been made, is not considered owing to the uncertainty in the determination of ξ_2 ; ξ_2 would have to be of the order 10,000.

examine whether a more general treatment of the formation of absorption lines in a stratified atmosphere leads to different results. We shall see that this is not the case.

Following Milne, we shall assume a "Taylor-expansion" for B_ν, retaining only the first two terms :

$$B_\nu = a_\nu + b_\nu \tau, \tag{21}$$

where

$$a_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT_0} - 1}, \tag{22}$$

$$b_\nu = \frac{2h\nu^3}{c^2} \frac{e^{h\nu/kT_0}}{(e^{h\nu/kT_0} - 1)^2} \frac{h\nu}{kT_0^2} \left(\frac{dT}{d\tau}\right)_{\tau=0}; \qquad \left(\frac{dT}{d\tau}\right)_0 = \tfrac{3}{8} T_0. \tag{23}$$

In (22) and (23) T_0 represents the boundary temperature.

With the form (21) for B_ν the solution of the equations proceeds exactly as in § 5. We have now (as one easily verifies),

$$\mathfrak{J}_\nu = a_\nu + b_\nu \tau + \alpha e^{-q_1 \tau} + \beta e^{+q_1 \tau}, \qquad (\tau < \tau_0) \tag{24'}$$

$$\mathfrak{J}_\nu = a_\nu + b_\nu \tau + \gamma e^{-q_2 \tau}, \qquad (\tau \geqslant \tau_0) \tag{24''}$$

$$\tfrac{3}{4}(1 + \eta_1) F_\nu = b_\nu + q_1(\beta e^{q_1 \tau} - \alpha e^{-q_1 \tau}), \qquad (\tau < \tau_0) \tag{25'}$$

$$\tfrac{3}{4}(1 + \eta_2) F_\nu = b_\nu - q_2 \gamma e^{-q_2 \tau}. \qquad (\tau \geqslant \tau_0) \tag{25''}$$

The equations determining the constants α, β and γ are found to be :

$$\left.\begin{aligned} &\alpha(1 + \eta_1 + \tfrac{2}{3} q_1) + \beta(1 + \eta_1 - \tfrac{2}{3} q_1) + (1 + \eta_1) a_\nu - \tfrac{2}{3} b_\nu = 0, \\ &\alpha e^{-q_1 \tau_0} + \beta e^{q_1 \tau_0} - \gamma e^{-q_2 \tau_0} = 0, \\ &\alpha e^{-q_1 \tau_0}(1 + \eta_2) q_1 - \beta e^{q_1 \tau_0}(1 + \eta_2) q_1 - \gamma e^{-q_2 \tau_0}(1 + \eta_1) q_2 + b_\nu(\eta_1 - \eta_2) = 0. \end{aligned}\right\} \tag{26}$$

The above equations can be solved for α and β, and we find for the emergent flux the expression

$$F_\nu(0) = \frac{4}{3} \frac{1}{\xi_1} \left\{ b_\nu + q_1 \frac{2\xi_1 b_\nu (\xi_1 - \xi_2) + (\xi_1 a_\nu - \tfrac{2}{3} b_\nu)[(\xi_2 q_1 + \xi_1 q_2) e^{q_1 \tau_0} - (\xi_2 q_1 - \xi_1 q_2) e^{-q_1 \tau_0}]}{(\xi_2 q_1 - \xi_1 q_2)(\xi_1 - \tfrac{2}{3} q_1) e^{-q_1 \tau_0} + (\xi_2 q_1 + \xi_1 q_2)(\xi_1 + \tfrac{2}{3} q_1) e^{q_1 \tau_0}} \right\}. \tag{27}$$

The emergent flux $F(0)$ in the immediate neighbourhood of the line (but in the continuous background) is obtained from (27) by putting $\eta_1 = \eta_2 = 0$ and $q_1 = q_2 = \sqrt{3}$. We obtain

$$F(0) = \frac{4a_\nu}{3} \frac{3 + \frac{b_\nu}{a_\nu}\sqrt{3}}{2 + \sqrt{3}}. \tag{28}$$

The residual intensity then is given by

$$r_\nu(\xi_1 ; \xi_2) = \frac{F_\nu(0)}{F(0)} = \frac{2 + \sqrt{3}}{\xi_1 \left(3 + \frac{b_\nu}{a_\nu}\sqrt{3}\right)}$$

$$\times \left\{ \frac{b_\nu}{a_\nu} + q_1 \frac{2\frac{b_\nu}{a_\nu}\xi_1(\xi_1 - \xi_2) + \left(\xi_1 - \frac{2}{3}\frac{b_\nu}{a_\nu}\right)[(\xi_2 q_1 + \xi_1 q_2) e^{q_1 \tau_0} - (\xi_2 q_1 - \xi_1 q_2) e^{-q_1 \tau_0}]}{(\xi_2 q_1 - \xi_1 q_2)(\xi_1 - \tfrac{2}{3} q_1) e^{-q_1 \tau_0} + (\xi_2 q_1 + \xi_1 q_2)(\xi_1 + \tfrac{2}{3} q_1) e^{q_1 \tau_0}} \right\}. \tag{29}$$

When $\xi_1 = \xi_2$, (29) reduces to

$$r_\nu(\xi_1 = \xi_2) = \frac{b_\nu + a_\nu q}{b_\nu + a_\nu \sqrt{3}} \cdot \frac{1 + \frac{2}{\sqrt{3}}}{1 + \eta + \frac{2}{3}q}. \tag{30}$$

From (29) we see that

$$r_\nu(\xi_1;\ 1) = \frac{2 + \sqrt{3}}{\xi_1\left(3 + \frac{b_\nu}{a_\nu}\sqrt{3}\right)}$$
$$\times \left\{ \frac{b_\nu}{a_\nu} + q_1 \frac{2\frac{b_\nu}{a_\nu}\xi_1(\xi_1 - 1) + \left(\xi_1 - \frac{2}{3}\frac{b_\nu}{a_\nu}\right)[(q_1 + \sqrt{3}\xi_1)e^{q_1\tau_0} - (q_1 - \sqrt{3}\xi_1)e^{-q_1\tau_0}]}{(q_1 + \sqrt{3}\xi_1)(\frac{2}{3}q_1 + \xi_1)e^{q_1\tau_0} - (q_1 - \sqrt{3}\xi_1)(\frac{2}{3}q_1 - \xi_1)e^{-q_1\tau_0}} \right\}. \tag{31}$$

$$r_\nu(1;\ \xi_2) = \frac{2 + \sqrt{3}}{\left(3 + \frac{b_\nu}{a_\nu}\sqrt{3}\right)}$$
$$\times \left\{ \frac{b_\nu}{a_\nu} + \frac{\frac{b_\nu}{a_\nu}(1 - \xi_2) + \left(\frac{1}{2} - \frac{1}{3}\frac{b_\nu}{a_\nu}\right)[(\sqrt{3}\xi_2 + q_2)e^{\sqrt{3}\tau_0} - (\sqrt{3}\xi_2 - q_2)e^{-\sqrt{3}\tau_0}]}{\xi_2(1{\cdot}0774e^{\sqrt{3}\tau_0} - 0{\cdot}0774e^{-\sqrt{3}\tau_0}) + \sqrt{\xi_2}(1{\cdot}0774e^{\sqrt{3}\tau_0} + 0{\cdot}0774e^{-\sqrt{3}\tau_0})} \right\}. \tag{32}$$

Formulæ (29), (31) and (32) correspond to our earlier formulæ (17), (19) and (20). One sees that these newer formulæ reduce to our earlier ones for the case $b_\nu/a_\nu = 3/2$. The formal discussion on the basis of these equations is of course the same as before.

10. *Applications.*—In § 8 (9) we applied formula (17) to the case of the H line of Ca^+ appearing in the wing of H_ϵ. For the early A-type stars one easily verifies that the appropriate value for b_ν/a_ν is almost exactly $3/2$. But when $b_\nu/a_\nu = 3/2$, as we have already pointed out, (29) is *identical* with (17), and consequently the earlier discussion of this problem requires no modification.

However, when one considers the problem of the *O*II lines appearing in the wing of H_γ our earlier discussion requires some reconsideration. For, in the case of B-type stars, at about 4350 A. the value of b_ν/a_ν is about 0·75. Hence for this case we have to use equations (29), (31) and (32) with $b_\nu/a_\nu = 0{\cdot}75$. Owing to the lengthy nature of the calculations required, a rough estimation has been made only for the *O*II line 4345·57 of θ Ophiuchi, assuming $\tau_0 = 1/6$. The roughly calculated central absorption is in fairly good agreement with the observations. Owing to the uncertainties of the measured intensities no further attempt has been made.

Summary

1. The distribution of the absorbing atoms of specified ionization and excitation is investigated as a function of the optical thickness τ of the stellar atmosphere. This problem may be important in spectral classification, in the dissociation equilibria of molecular compounds, in the

investigation of stars of the ξ Aurigæ type, and especially in the interpretation of an effect observed by Swings and Struve. These authors have found that absorption lines appearing in the wings of other lines have peculiar intensities which cannot be interpreted by the usual formulæ for the absorption lines.

2. The cases of H and O^+ in B-type stars and of Ca^+ and H in A-type stars are investigated numerically; differences of distribution appear conspicuously; the effect of surface gravity is important.

3. The conditions of radiative equilibrium in a sharply stratified atmosphere are investigated; an adequate formula is provided for the intensities of the corresponding absorption lines.

4. This formula explains the measures by Swings and Struve, when it is assumed that:

(*a*) in the examined A-type stars, the Ca^+ absorbing atoms occupy the upper part of the reversing layers, until approximately $\tau_0 = \frac{1}{6}$;

(*b*) in the B-type stars, for which the calculation has been made, the absorbing hydrogen atoms are in the upper part of the atmosphere until $\tau_0 = \frac{1}{6}$; the absorbing ionized oxygen atoms would be below $\tau_0 = \frac{1}{6}$.

Roughly speaking, this would mean that the effect of the different distributions of absorbing atoms is somewhat similar to that of a sharp stratification.

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DYNAMICS OF RADIATION PRESSURE FOR A DIFFUSE NEBULA.

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1. *Introduction and Summary.*—The importance of L_α radiation pressure for nebulæ was first pointed out by Ambarzumian.* He gave a complete solution of the radiative equilibrium in a stationary nebula, using the Schwarzschild-Schuster approximation. A closer approximation allowing the intensity to vary with direction was presented by Chandrasekhar.†

According to well-known mechanisms, line spectra are produced in a nebula under the action of the exciting star. Among these the first line L_α of the Lyman series of hydrogen is most vigorously absorbed, and the resulting radiation pressure tends to "blow up" the nebula, as Ambarzumian showed. As soon as the nebula starts expanding, however, the radiation pressure is greatly reduced, since the Doppler effect of the moving material makes it permeable to L_α radiation from the more inward parts. The author,‡ modifying Ambarzumian's methods, worked out the case of radiative

* V. A. Ambarzumian, *M.N.*, **93**, 50, 1932.
† S. Chandrasekhar, *Zeitschr. f. Astrophys.*, **9**, 266, 1935.
‡ H. Zanstra, *M.N.*, **95**, 84, 1934 (Comm. Univ. Obs. Oxford, No. 2).

equilibrium for an expanding nebula for which the velocity v is proportional to the optical depth τ, and showed that the radiation pressure was indeed greatly reduced, provided the thermal width was small compared with the velocity of expansion. In this same paper the procedure to be followed for a general type of expansion where v is any function of τ was also outlined. It forms the basis of the present considerations and will be referred to as paper I. McCrea and Mitra * have made progress towards the solution of a similar problem, introducing the refinement that the intensity varies with direction. They confine themselves to the special case of pure scattering which is not directly applicable to nebulæ.

The present paper deals with two questions. In the first place the expression for the force of radiation pressure for a general distribution of v as a function of τ is brought into a simple form of a fairly high degree of approximation adapted to direct dynamical application.

In the second place, using this simple expression, the dynamics of a diffuse nebula is worked out, and the theoretical result is compared with the available observations for the Orion Nebula. Suppose a large cloud of nebular material as presented by the Orion Nebula is illuminated by a high-temperature star in its neighbourhood. It is clear that stellar radiation will be absorbed at the surface of the nebular cloud and will be emitted as a line spectrum in accordance with current theory. But what will happen dynamically? If the nebula were not sufficiently opaque to stellar radiation, it seems probable that the L_α radiation produced in the nebular atoms at relative rest would exert a momentum in all directions, and the nebula would tend to "blow up." Suppose, however, the nebula to be sufficiently opaque, so that the radiation cannot get through, then the L_α quanta can only escape in the direction towards the star, and consequently the net flux and the radiation pressure due to the L_α will be directed towards the star. We should therefore expect on these grounds that the nebular material will be driven towards the star, this process continuing until the nebular material is sufficiently exhausted to become transparent, when the nebula dissipates. A large cloud like the Orion Nebula, which is mainly luminous in its outer portions, evidently presents the stage when the nebula is still opaque. It is our purpose to investigate the nature of the motion set up in this opaque state and its bearing on observable quantities, in particular the shape of the line emitted.

It is clear that on the present picture the line width should be proportional to the cosine of the angle which the nebular plane makes with the tangential plane to the celestial sphere. For this reason it would be of interest to observe cases of nebulæ seen edge on, where this line width might be expected to be abnormally small.

In order to make the detailed treatment in the body of the paper more easy to follow, the remaining sections are summarised below.

Section 2 gives a recapitulation of the formulæ of Paper I which form the starting-point for the present investigation.

In *Section* 3 two approximations are introduced. In the first place the

* W. H. McCrea and K. K. Mitra, *Zeitschr. f. Astrophys.*, **11**, 359, 1936.

quantity u, representing the ratio of the absorption coefficient of L_c to that of L_α, is put equal to zero in the general expression (10) for the radiation force, which introduces an error of only 10^{-7}. In the second place the optical depth τ of L_c near the point $\bar{\tau}$, where this force is determined, is developed as a function of the velocity with respect to this point $(v-\bar{v})$ in the form of a Taylor series, of which the first three powers of $(v-\bar{v})$ are retained. An approximate expression for the force of radiation pressure is then obtained which involves only the first and second derivatives of τ with respect to v and has a relative accuracy of about 1 per cent. for practical cases of nebulæ.

In *Section* 4 the dynamics of a plane nebula of the type just described are developed, and by means of the equation of continuity (18) the v is explicitly expressed as a function of τ by equation (25), involving τ_0', the derivative of τ with respect to v at the point $\tau=0$ where the star is situated, and a constant A defined by (23). For the simple case $A << \tau_0'^3$, the distribution of velocities (fig. 1*a*) and of concentrations in space and the luminosity distributions (fig. 1*b*) are obtained, using the ionisation formula (16) (see p. 44). The luminosity distribution in v gives a triangular shape of the spectral line (fig. 2) for the nebula viewed perpendicularly (see p. 54).

Section 5 deals first with the case $A >> \tau_0'^3$, where the various distributions are also given in the figures mentioned. The spectral line (fig. 2) shows a nearly symmetrical shape. The general case is then treated, and it is shown that the physical situation is entirely determined by the quantity $-v_0\tau_0'$ which may vary from 2, the case $A << \tau_0'^3$, to 0, the case $A >> \tau_0'^3$. For this general case the formulæ governing the distributions are also given, and for $-v_0\tau_0'=1$ and 1·5 the shape of the spectral lines is calculated (fig. 2).

In *Section* 6 Trumpler's value of the parallax of the Orion Nebula and Hopmann's photometry and distance determinations in this nebula fix all the quantities necessary for describing the state of motion and physical characteristics, except the constant $-v_0\tau_0'$. Since observations seem to indicate a fairly symmetrical spectral line, it is assumed that $A >> \tau_0'^3$, viz. $-v_0\tau_0'=0$. Then the width of the spectral lines of hydrogen for which the intensity is 0·7 of its maximum value comes out to 14 km./sec. The ionisation formula used, however, is not entirely correct and the actual concentration of ionised atoms might well be ten times lower than the theoretical, which would give a width of 30 km./sec. The observed width by Baade, Goos, Koch and Minkowski is 25 km./sec., and hence there appears to be approximate agreement between observation and theory.

2. *Results of Previous Investigation.*—In the present work we shall make use of the following results from paper I. The nebula is approximated by a portion of a plane slab at a distance r from the exciting star and assumed to consist mainly of hydrogen.

The incident flux πS_c representing the number of ultra-violet quanta L_c per cm.^{2} per sec. able to ionise the hydrogen is then given by

$$S_c=\frac{2R^2}{r^2}\frac{k^3T^3}{c^2h^3}\int_{x_0}^{\infty}\frac{x^2}{e^x-1}dx, \qquad x_0=\frac{h\nu_0}{kT}, \tag{1}$$

where ν_0 is the ionising frequency, R the radius and T the black-body temperature of the star. For the absorption coefficient of this L_c radiation beyond the head of the Lyman series its value at the head is adopted, viz. $\alpha_c = 0{\cdot}54 \times 10^{-17}$ per H-atom. Then the optical depth for this radiation is

$$\tau = \int_0^{\xi} \alpha_c C_a d\xi, \tag{2}$$

where ξ is the depth in the nebula in the direction *away* from the star and C_a the number of normal H-atoms per cm.^{3}.

For every quantum of L_c radiation absorbed in the nebula, one quantum of radiation in the first line of the Lyman series L_α is ultimately re-emitted. For the sake of simplicity this re-emission is considered to take place on the spot. Now this emission line is much more vigorously absorbed than the original continuous spectrum at the head of the series, and the line radiation L_α is the predominant factor governing the radiation pressure in the nebula.

Consider a volume element of the nebula viewed perpendicularly to the slab. Then the atoms within this element have velocities of agitation or "thermal" velocities W in this direction. The actual situation is now approximated by assuming that the thermal velocities W are equally frequent in the interval from $-w$ to $+w$. This amounts to replacing the actual spectral line by one of rectangular contour with a total thermal width of $2w$. Within this spectral line the absorption coefficient of L_α has the value α_l. Its ratio to the absorption coefficient of L_c is given by

$$u = \frac{\tau}{\sigma} = \frac{\alpha_c}{\alpha_l} = \mu w, \tag{3}$$

$$\mu = 0{\cdot}81 \times 10^{-5} \text{ (km./sec.)}^{-1} = 0{\cdot}81 \times 10^{-10} \text{ (cm./sec.)}^{-1}. \tag{3a}$$

By analogy with (2), the formal optical depth σ for L_α is introduced

$$\sigma = \int_0^{\xi} \alpha_l C_a d\xi. \tag{4}$$

In an expanding nebula this may not be a true optical depth, since due to the Doppler effect only a limited portion of the nebula is active for the line absorption of a certain frequency.

When no specific statement is made, the term "velocity" is used for the velocity component perpendicular to the slab. By the mechanical velocity v is meant the velocity of matter in bulk, viz. of a volume element containing a large number of particles. The most general kind of expansion at a certain instant is characterised by

$$v = f(\tau), \tag{5}$$

since then the velocity may vary from point to point in an arbitrary manner. This equation may also be written

$$\tau = \phi(v). \tag{6}$$

The velocity of an individual atom V is the resultant of its thermal velocity W in the volume element and the mechanical velocity of the volume element, hence

$$V = v + W. \tag{7}$$

Using the Schwarzschild-Schuster approximation, the equations of radiative transfer were now solved in paper I for the group of atoms having atomic velocities in the interval $V - \frac{1}{2}dV$ to $V + \frac{1}{2}dV$, which were shown to occur only at optical depths between τ_a and τ_b given by

$$\tau_a = \phi(V - w), \qquad \tau_b = \phi(V + w). \tag{8}$$

After integration over all the atoms in one volume element, the force of radiation pressure per cm.^{2} on an interval $d\tau$ at the optical depth $\bar{\tau}$ was shown to be

$$dP_r = \frac{h\nu_a}{c}\frac{d\tau}{u}\pi F_{\text{eff}}, \tag{9}$$

where the effective flux πF_{eff} was given by *

$$F_{\text{eff}} = \frac{S_c}{2w}\int_{W=-w}^{+w}\left[-e^{-\bar{\tau}} + \frac{(e^{-\tau_a} - e^{-\tau_b}) + \frac{1}{2}u(e^{-a\tau} + e^{-\tau_b})}{u + \tau_b - \tau_a}\right]dW. \tag{10}$$

Though this was not explicitly stated, the equation (10) was obtained without any special assumption as to the functions f or ϕ. Hence it applies to this most general type of expansion. It should be understood, however, that the equations do not apply for the outer and inner edge of the nebula, where the mechanical velocity v differs from its value at the boundary by less than w.

3. *Force of Radiation Pressure for General Type of Expansion.*—From equation (8) we have, as regards order of magnitude, $\tau_b - \tau_a \simeq 2wd\phi/dv = 2wd\tau/dv$. For a spectral line of a planetary or diffuse nebula an appreciable portion of the luminosity is due to optical depths τ from 0 to 1, so that the width of the spectral line is represented by the range of velocity v_L corresponding to this range of τ. Hence, as regards order of magnitude, $d\tau/dv = 1/v_L$ and $\tau_b - \tau_a \simeq 2w/v_L$ or the ratio of the thermal width to the observed width of the spectral line. Now for planetary nebulæ it was argued in paper I that this ratio must be much smaller than 1, since only then the acceleration due to radiation pressure is small enough to give agreement with the observed line widths. The same thing may be said for the diffuse nebulæ, or for the Orion Nebula at any rate, since a purely mechanical interpretation may account for the observed line width of O_{III} and O_{II} as compared with H. For planetary and diffuse nebulæ both the value $\tau_b - \tau_a \simeq 0{\cdot}1$ seems a fair estimate of the order of magnitude.

Taking v_L of the order 20 km./sec., we get w of the order 1 km./sec., so that, using (3), u is of the order 10^{-5}. Because of this extreme smallness,

* Equation (40) in paper I contains a misprint. The τ_b in the denominator should have a + sign, as in the present equation (10). The former equation (36) with $d\sigma = d\tau/u$ (3) is the present equation (9).

we obtain a good approximation if we put $u=0$ in (10). The result is

$$F_{\text{eff}}e^{+\bar{\tau}}S_c^{-1}=\int_{W=-w}^{+w}\frac{1}{2}\left[-1+\frac{e^{-(\tau_a-\bar{\tau})}-e^{-(\tau_b-\bar{\tau})}}{\tau_b-\tau_a}\right]\frac{dW}{w}, \tag{11}$$

the error in the integral being the larger of the two quantities $\frac{1}{12}u(\tau_b-\tau_a)$ and u^3, as may be seen by putting $e^{-\tau_a}$ outside the fraction in the integrand of (10) and expanding. Considering the above orders of magnitude, the error is only 10^{-7}.

In paper I we calculated the radiation pressure when the mechanical velocity v was proportional to τ, so that we could use the expression $\tau-\bar{\tau}=a(v-\bar{v})$.

We now wish to solve the problem for a general type of expansion (5), (6), so that v may be any function of τ. To this end we consider the function ϕ (6) for optical depths τ near $\bar{\tau}$ developed as a Taylor series, of which we take powers up to the third into consideration.

$$\tau=\bar{\tau}+a_1(v-\bar{v})+a_2{}^2(v-\bar{v})^2+a_3{}^3(v-\bar{v})^3, \tag{12}$$

where

$$a_1=\frac{d\tau}{dv}, \qquad 2a_2{}^2=\frac{d^2\tau}{dv^2}, \qquad 6a_3{}^3=\frac{d^3\tau}{dv^3}, \tag{13}$$

the derivatives being taken at $\tau=\bar{\tau}$, $v=\bar{v}$, that is for the point where we determine the radiation pressure according to (9) and (11). Considering $\tau-\bar{\tau}$ as a small quantity of the first order, the term $a_1(v-\bar{v})$ is likewise of the first order, $a_2{}^2(v-\bar{v})^2$ is of the second and $a_3{}^3(v-\bar{v})^3$ of the third order. So with the notation used the power of the coefficient a indicates also the order of the quantity.* It should be remarked that $a_2{}^2$ in (12) can, of course, be negative and the a_2 imaginary ; however, it always occurs as a square, so that this is no objection.

Using (12) for $\phi(v)$ (6), the equation (8) gives

$$\tau_a=\bar{\tau}+a_1(V-w-\bar{v})+a_2{}^2(V-w-\bar{v})^2+a_3{}^3(V-w-\bar{v})^3,$$

and $\tau_b=$the same expression with $+w$ for $-w$. Now since W in (10) is the thermal velocity of an atom in the volume element considered at $\tau=\bar{\tau}$, $v=\bar{v}$, we have from (7) $V=\bar{v}+W$, so that

$$\left.\begin{aligned}\tau_a-\bar{\tau}&=a_1(W-w)+a_2{}^2(W-w)^2+a_3{}^3(W-w)^3\\ \tau_b-\bar{\tau}&=a_1(W+w)+a_2{}^2(W+w)^2+a_3{}^3(W+w)^3\\ \tau_b-\tau_a&=2a_1w+4a_2{}^2wW+2a_3{}^3(w^3+3wW^2)\end{aligned}\right\} \tag{14}$$

to be substituted in (11). We have seen above that $\tau_b-\tau_a\simeq 0{\cdot}1$, and since $w>W>-w$, $\tau_a-\bar{\tau}$, and $\tau_b-\bar{\tau}$ are also of this order, so our first order quantities are $\simeq 0{\cdot}1$, whereas quantities with a coefficient a^n are $\simeq(0{\cdot}1)^n$.

In accordance with the expansion (12), we wish to obtain the integral (11) to third order of small quantities. To this end we should develop the

* The reader will find this notation useful in actually carrying out the expansion of (10). For this purpose it is also convenient to introduce the three first-order quantities $\alpha_1=a_1w$, $\alpha_2=a_2w$, $\alpha_3=a_3w$ and the quantity $x=W/w$ of order unity.

numerator of the fraction occurring in (11) to the 4th power. The integral then reduces to

$$\int_{W=-w}^{+w} \tfrac{1}{2}(\tau_b - \tau_a)^{-1} \sum_{n=2}^{4} \frac{1}{n!}[(\tau_a - \bar{\tau})^n - (\tau_b - \bar{\tau})^n] \frac{dW}{w},$$

where we remark that the -1 has cancelled against the first term under the summation sign, so that Σ starts with $n = 2$.

Substituting (14), and retaining only quantities of the first three orders in the result, we obtain upon integration

$$F_{\text{eff}} e^{+\tau} S_c^{-1} = \tfrac{1}{3} a_1^2 w^2 - \tfrac{4}{3} a_2^2 w^2, \qquad (15)$$

the error in which is of the 4th order, viz. $a_1^4 w^4$ or $(0{\cdot}1)^4$. The relative accuracy is $a_1^2 w^2$, so that (15) gives F_{eff} with an accuracy of about 1 per cent. in the practical case of diffuse and planetary nebulæ. Since henceforth we are indicating the point in the nebula where the radiation pressure is considered by τ, v, instead of by $\bar{\tau}$, $\bar{v}$, we have put τ instead of $\bar{\tau}$ in the result. The meaning of the coefficients a_1 and a_2^2 occurring in (15) is therefore given by the equations (13) as they stand.

The rather unexpected features about the simple result (15) are that the coefficient a_1 does not occur to the first power, that no mixed coefficients like $a_1 a_2^2$ occur and that the a_3^3 is entirely absent, though the order of approximation is the third. This is all due to the fact that $\int_{-w}^{+w} W^m dW$ for odd powers of W is zero: indeed, a_1 occurs in the integrand as $-a_1 W$, *e.g.* the mixed coefficients occur in terms like $2a_1 a_2^2 W$ and $2a_1 a_2^2 W^3$, and a_3^3 as $-a_3^3 W$, which all give zero upon integration. For the derivations which are to follow this circumstance is very fortunate, since it means that the force of radiation pressure is very well represented by the first and second derivatives of τ with respect to v, and that the third derivative is not at all required for giving a third-order approximation.

4. *Dynamics of a Plane Diffuse Nebula*: 1. *Case* $A << \tau_0'^3$.—We shall consider a plane diffuse nebula (fig. 1) as an infinite plane slab of very large (∞) concentration C (shaded portion in the figure) and velocity zero, from which material emerges towards the left with a velocity v which is a function of τ. The observer is thought to be towards the left at a large distance, and the line of sight makes an angle ϕ with the normal to the nebula. This model is suggested by our mechanical interpretation of line widths in the Orion Nebula given in paper I, section 8. It is further assumed that a stationary state is set up, so that this function does not depend on the time. Throughout the paper we shall keep the same notation as in the previous sections: the incident flux S_c comes from the left, the optical depth τ and co-ordinate ξ are reckoned towards the right, hence $v = d\xi/dt$ for an element of moving matter is also positive towards the right, and the actual velocities we are dealing with, v as well as v_0, are all negative quantities.

The plane where the concentration becomes infinite will be termed "plane of the nebula." From this plane the material is removed and made to flow out by the very action of radiation pressure. Now the radiation

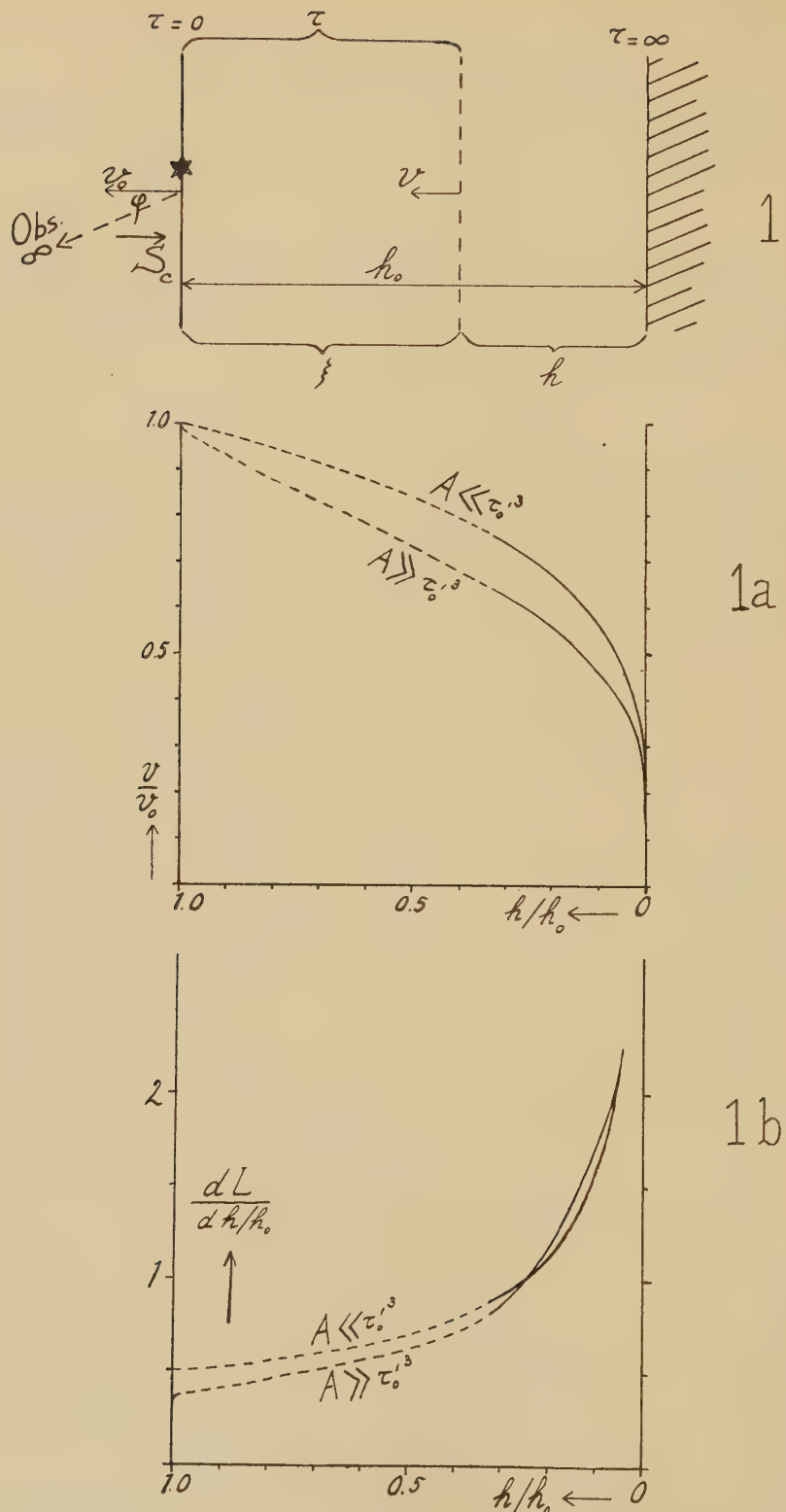

FIG. 1.—*Model of a diffuse nebula* : (*a*) *Distribution of velocity as function of the distance to the nebula, both expressed in their value at* $\tau = 0$; (*b*) *distribution of luminosity as function of the relative distance* h/h_0 *to the nebula.*

pressure acts for all values of τ up to infinity, hence the plane of the nebula must correspond to $\tau = \infty$. The point $\tau = 0$ is where the radiation is incident on the outflowing material, viz. the point itself where the star is situated. Hence the total depth h_0 represents the distance of the star to the nebula, whereas S_c may be taken to represent the flux of stellar radiation at the nebular surface if there were no absorption, viz. (1) with $r = h_0$. This treatment of reducing the problem to one of parallel slabs is only satisfactory when the main absorption takes place near the plane $\tau = \infty$, and the solution thus obtained can only be trusted for distances h near to this plane, say up to $h = \frac{1}{3}h_0$.

It may further be remarked that the more general case where the direction of S_c makes an angle θ with the normal to the nebular plane is obtained by taking everywhere $S_c \sec\theta$, $\alpha_c \sec\theta$, $\tau \sec\theta$ and $u \sec\theta$ instead of S_c, α_c, τ and u, since the generation of quanta inside the nebula is the same as would be produced by a normal flux $\pi S_c \sec\theta$ and an absorption coefficient $\alpha_c \sec\theta$ of L_c, whereas the line absorption coefficient does not change.

The ionisation formula is * $C_a = \gamma C_i^2 e^{+\tau}$, where C_a and C_i are numbers of normal H-atoms and of ions or electrons per cm.^{3}. For nebulæ the assumption that the ionisation is nearly complete seems justified, except for very large τ, so that to a fair order of approximation

$$C_a = \gamma C^2 e^{+\tau}, \tag{16}$$

where C is the total number of H-atoms, normal and ionised, per cm.^{3} and

$$\gamma = 4\frac{r^2}{R^2}\frac{e^{\frac{157200}{T}}}{1{\cdot}222 \times 10^{15} T^{3/2}}, \tag{17}$$

using the same notation as for formula (1). The factor γ takes care of the dilution, and the factor $e^{+\tau}$ of the absorption of the ionising radiation. We shall use the formula (16) for any value of τ. When in this formula $\tau \to \infty$, $C_a \to \infty$, even when C is finite, which of course is a physical impossibility, since C_a can never exceed C. Physically it means that there the concentration C_a is very large compared with its value at $\tau = 0$ and that the absorption of L_c becomes practically complete.

The assumption of a stationary state leads to the equation of continuity

$$Cv = C_0 v_0, \tag{18}$$

where C_0 the concentration at $\tau = 0$. It states that the total number of atoms passing per second through any plane parallel to the slab is the same and independent of the time.

We shall now discuss the dynamics of this simple model of a plane diffuse nebula. Consider 1 cm.^{3} of nebular material of cross-section 1 cm.^{2} parallel to the plane of the nebula and of a depth $\Delta\xi = 1$ cm. perpendicular to it. To the latter corresponds an optical depth (2) $\Delta\tau = \alpha_c C_a$. The

* *M.N.*, **93**, 131, 1932, formula (19).

force of radiation pressure on this 1 cm.^{3} becomes therefore, according to (9), (15), (3) and (13),

$$\Delta P_r = \frac{h\nu_a}{c}\frac{\alpha_c C_a}{\mu w}\pi S_c e^{-\tau}\left[\frac{1}{3}\left(\frac{d\tau}{dv}\right)^2 - \frac{2}{3}\frac{d^2\tau}{dv^2}\right]w^2. \tag{19}$$

Since a stationary state is assumed, v is a function of τ which does not depend on the time, so that the acceleration is

$$\frac{dv}{dt} = \frac{dv}{d\xi}\frac{d\xi}{dt} = v\frac{dv}{d\xi}. \tag{20}$$

Multiplying this by the mass Cm_H per cm.^{3}, where m_H is the mass of the hydrogen atom, the force becomes, on account of (2),

$$\Delta P_r = m_H C_0 v_0 \alpha_c C_a \frac{dv}{d\tau}, \tag{21}$$

where the equation of continuity (18) has served to replace Cv by $C_0 v_0$.

Putting the two expressions for ΔP_r (19) and (21) equal to each other and considering that

$$\frac{d^2v}{d\tau^2} = -\frac{d^2\tau}{dv^2}\Big/\left(\frac{d\tau}{dv}\right)^3,$$

we obtain the differential equation for v as a function of τ

$$\frac{d^2v}{d\tau^2} + \frac{1}{2}\frac{dv}{d\tau} = -\frac{A}{6}e^{+\tau}\left(\frac{dv}{d\tau}\right)^4, \tag{22}$$

where

$$A = -\tfrac{9}{2}\mu\frac{m_H c}{h\nu_a}\frac{C_0 v_0}{S_c w}. \tag{23}$$

Velocities are directed towards the left and therefore negative, so that A is a positive quantity. It has the dimension (velocity)$^{-3}$.

Introducing $\left(\frac{d\tau}{dv}\right)^3$ as independent variable, equation (22) reduces to a linear differential equation which has the solution

$$\left(\frac{d\tau}{dv}\right)^3 = -Ae^{+\tau} + (\tau_0'^3 + A)e^{+\frac{3}{2}\tau}, \tag{24}$$

where the constant τ_0' may be interpreted as the value of $\frac{d\tau}{dv}$ for $\tau = 0$.

Multiplying both sides by $e^{-\frac{3}{2}\tau}$ and introducing $e^{-\frac{1}{2}\tau}$ as independent variable, the equation becomes separable and has the solution

$$v = \frac{3}{A^{1/3}}\left[\left(\frac{\tau_0'^3}{A} + 1 - e^{-\frac{1}{2}\tau}\right)^{2/3} - \left(\frac{\tau_0'^3}{A} + 1\right)^{2/3}\right], \tag{25}$$

satisfying the boundary condition that the plane of the nebula $\tau = \infty$ is at rest.

The distribution of v as a function of τ (25), together with the ionisation formula (16) and the equation of continuity (18) provides the means of

obtaining a complete description of the physical situation. When A is very much smaller than ${\tau_0}'^3$, this description is a very simple one. This case is therefore considered first, whereas more complicated cases are treated in the next section.

Case $A << {\tau_0}'^3$.—If this condition be fulfilled, we have the approximation

$$\left(\frac{{\tau_0}'^3}{A}+1-e^{-\frac{1}{2}\tau}\right)^{2/3}=\left(\frac{{\tau_0}'^3}{A}+1\right)^{2/3}-\frac{2}{3}\left(\frac{{\tau_0}'^3}{A}+1\right)^{-1/3}e^{-\frac{1}{2}\tau}$$

and from (25), applying the condition once more, $v=-2e^{-\frac{1}{2}\tau}/{\tau_0}'$, and hence

$$v=v_0e^{-\frac{1}{2}\tau}, \tag{26}$$

where

$$v_0=-\frac{2}{{\tau_0}'}. \tag{27}$$

Like v (26), the quantities C_a and C are expressible as functions of τ alone. Equations (26), (16) and (18) give

$$C=C_0e^{+\frac{1}{2}\tau}, \qquad C_a=\gamma {C_0}^2e^{+2\tau}. \tag{28}$$

In its turn, τ is expressible as a function of ξ by the differential equation

$$\frac{d\tau}{d\xi}=\alpha_c\gamma {C_0}^2e^{+2\tau}, \tag{29}$$

from the definition (2) of τ and expression (28) for C_a.

Integrating and introducing the distance h from the plane $\tau=\infty$ instead of ξ, we obtain

$$e^{-\tau}=\left(\frac{h}{h_0}\right)^{1/2}, \tag{30}$$

$$h_0=\frac{1}{2\gamma\alpha_c{C_0}^2}, \tag{30a}$$

where h_0 is also the depth of the emerging material from the plane $\tau=0$, where the star is situated, to the plane of the nebula $\tau=\infty$.

Substituting (30) in (26) and (28), we obtain the *space distribution of velocities and concentrations*

$$\frac{v}{v_0}=\left(\frac{h}{h_0}\right)^{1/4}, \qquad C=\frac{1}{(2\gamma\alpha_ch_0)^{1/2}}\left(\frac{h_0}{h}\right)^{1/4}, \qquad C_a=\frac{1}{2\alpha_ch}, \tag{31}$$

where (30*a*) has served to replace C_0 by h_0.

The curve $A << {\tau_0}'^3$ of fig. 1*a* gives the value of v/v_0, viz. the velocity of the outflowing material expressed in its maximum value as unit, as a function of h/h_0, viz. the distance from the nebula expressed in the distance of the star to the nebula as unit, as obtained from (31). The untrustworthy points beyond $h/h_0=1/3$ are dotted. The nebular surface is considered at the right, so that h/h_0 is plotted towards the left.

According to the theory of nebular luminosity, the number of quanta emitted as lines of the Balmer series or as the continuous spectrum at its head is approximately equal to the number of quanta L_c absorbed. Hence the

fraction of luminosity emitted in the observable hydrogen spectrum between the optical depths ∞ and τ is

$$L = e^{-\tau}, \tag{32}$$

since the simplification has been introduced that the conversion of L_c into Ba and Ba_c takes place on the spot (section 2).

The equations (30) and (32) give the *luminosity distribution* as function of the *relative distance* to the nebula

$$L = \left(\frac{h}{h_0}\right)^{1/2}, \qquad \frac{dL}{dh/h_0} = \tfrac{1}{2}\left(\frac{h_0}{h}\right)^{1/2}. \tag{33}$$

According to this, one-half of the luminosity of hydrogen would be concentrated within $h = \frac{1}{4}h_0$. The curve $A << \tau_0'^3$ of fig. 1*b* gives the dependence of $dL/d(h/h_0)$ on h/h_0, plotted towards the left, the area under the curve representing the fraction of luminosity in a certain interval of h/h_0.

Similarly (26) and (32) give

$$L = \left(\frac{v}{v_0}\right)^2, \qquad \frac{dL}{dv/v_0} = 2\frac{v}{v_0}. \tag{34}$$

The second equation (34) gives the *relative shape of the spectral line* for a line of sight perpendicular to the plane of the nebula. The intensity, starting with zero at the red end, increases linearly towards the violet. This is shown by the curve $A << \tau_0'^3$ of fig. 2, where v/v_0 is plotted towards the left and the uncertain portion beyond $h = \frac{1}{3}h_0$, viz. $v = 0{\cdot}76\ v_0$, is dotted.

5. *Dynamics of a Plane Diffuse Nebula: 2. Case* $A >> \tau_0'^3$. *3. General Case.*

Case $A >> \tau_0'^3$.—For this case we shall use the *abbreviation*

$$y = (1 - e^{-\frac{1}{2}\tau})^{2/3}, \tag{35}$$

whenever the expressions would otherwise become complicated.

Since we may put $\tau_0'^3/A = 0$, the equation (25) reduces to

$$v = v_0[1 - (1 - e^{-\frac{1}{2}\tau})^{2/3}] = v_0(1 - y), \tag{36}$$

where

$$v_0 = -\frac{3}{A^{1/3}}, \tag{37}$$

since for $\tau = 0$, $v = v_0$.

Now the A is defined by (23) and contains the v_0 also. Introducing this definition (23), we may solve v_0 from (37)

$$v_0^4 = \frac{2}{3}\frac{h\nu_\alpha}{m_H c\mu}\frac{S_c w}{C_0}. \tag{38}$$

The equation of continuity (18) with (36) gives

$$C = \frac{C_0}{1 - y}. \tag{39}$$

Substituting (39) in the ionisation formula (16), we obtain

$$C_a = \frac{\gamma C_0^2}{(1-y)^2 e^{-\tau}}. \tag{40}$$

From the definition (2) of τ, we have

$$\alpha_c d\xi = \frac{d\tau}{C_a}, \tag{41}$$

and substituting (40),

$$\alpha_c \gamma C_0^2 \xi = \int_{\tau=0}^{\tau} e^{-\tau} d\tau + 2\int_{\tau=0}^{\tau} y de^{-\tau} - \int_{\tau=0}^{\tau} y^2 de^{-\tau} \tag{42}$$

Expressing $e^{-\tau}$ in terms of y by (35), the second and third integral are readily obtained, and the result is

$$\alpha_c \gamma C_0^2 \xi = 1 - e^{-\tau} - \tfrac{12}{5} y^{5/2} + \tfrac{6}{7} y^{7/2} + \tfrac{3}{2} y^4 - \tfrac{3}{5} y^5. \tag{43}$$

For $\tau = \infty$, $\xi = h_0$ the geometrical depth of the outflowing material, which is interpreted as the distance of the star from the nebula. Hence (43) and (35) give

$$\alpha_c \gamma C_0^2 h_0 = \tfrac{5}{14}, \tag{44}$$

whereas the corresponding value of the equation (30*a*) for the case $A << \tau_0'^3$ is $\frac{1}{2}$. The distance of any point to the nebula is $h = h_0 - \xi$, hence (43) and (44) give

$$\frac{h}{h_0} = 1 - \tfrac{14}{5}(1 - e^{-\tau} - \tfrac{12}{5} y^{5/2} + \tfrac{6}{7} y^{7/2} + \tfrac{3}{2} y^4 - \tfrac{3}{5} y^5). \tag{45}$$

The problem is now completely solved. For a star of given radius R and temperature T at a given distance h_0 from the nebula, C_0 is determined by (44), and then v_0 by (38). Then, for any assigned optical depth τ, or y (35), the distance h from the nebula follows from (45), whereas v, C and C_a follow from (36), (39) and (40). Hence all quantities are expressible as functions of h by means of the parameter τ.

The curve $A >> \tau_0'^3$ of fig. 1*a* represents the v/v_0 thus obtained as a function of h/h_0, plotted towards the left. As compared with the upper curve for $A << \tau_0'^3$, the v/v_0 first increases less rapidly with h/h_0, but near $h = h_0$ the increase becomes very rapid, and at $h = h_0$ the curve has a vertical tangent, as is best seen by developing (36) and (45) for small τ. At this point the acceleration becomes infinite, since (36) $\frac{d^2\tau}{dv^2} = \infty$, so that (19) $\Delta P_r = -\infty$.

From (32) we obtain

$$\frac{dL}{dv/v_0} = v_0 \frac{dL}{d\tau}\frac{d\tau}{dv} = -e^{-\tau} v_0 \frac{d\tau}{dv}, \tag{46}$$

which also holds for the most general case.

Differentiation of (36) with respect to τ and (46) give

$$\frac{dL}{dv/v_0} = 3(1 - e^{-\frac{1}{2}\tau})^{1/3} e^{-\frac{1}{2}\tau}. \tag{47}$$

The equations (47) and (36) give the *relative shape of the spectral line* for the nebula viewed perpendicularly, expressed by means of the parameter τ. It is represented by the curve $A >> \tau_0'^3$ in fig. 2, and the less certain portion beyond $h = \frac{1}{3}h_0$, corresponding to $v/v_0 = 0{\cdot}65$, is dotted. The reason why the curve shows a maximum is that, as the point $\tau = 0$ is approached, the acceleration increases rapidly, so that the spreading out of the material for the same velocity range more than offsets the increase in luminosity due to the stronger L_c. At $\tau = 0$ the acceleration becomes infinite, and dL/dv becomes zero. It is further remarkable that the spectral line is nearly symmetrical.

From (32) and (35) follows

$$\frac{dy}{dL} = -\tfrac{1}{3}e^{+\frac{1}{2}\tau}y^{-\frac{1}{2}}. \tag{48}$$

Differentiating (45) with respect to L and using (48) and (32), we obtain the equation

$$\frac{dL}{dh/h_0} = \tfrac{5}{14}[1 + e^{+\frac{1}{2}\tau}(-2y + y^2 + 2y^{5/2} - y^{7/2})]^{-1}. \tag{49}$$

Together with (45) it expresses the *luminosity distribution* as function of the *relative distance* to the nebula. It is represented by the curve $A >> \tau_0'^3$ in fig. 1*b*. The luminosity shows a rather strong concentration towards the nebular surface. One-half of it is concentrated within $h = 0{\cdot}20h_0$. Quite near $h = h_0$ the curve bends down, and it has a vertical tangent at this point. This is best seen by developing (49) and (45) for small τ.

The case $A >> \tau_0'^3$ has been given a separate treatment, because there are reasons for believing that it may be a good approximation for some actual cases.

General Case.—If we do not restrict ourselves as to the value of $\tau_0'^3/A$, the equation (25) represents a set of solutions each of which is characterised by this constant. It will be shown that, for a star of given properties at a given distance h_0 from the nebula, this constant $\tau_0'^3/A$ also completely determines the physical situation. In accordance with this we have to choose our formulation in such a way that all quantities become expressed in terms of $\tau_0'^3/A$, h_0, γ, S_c and w. The procedure is quite analogous to that followed for the case $A >> \tau_0'^3$.

Whenever it is desirable, we shall use the abbreviation

$$z = \left(\frac{\tau_0'^3}{A} + 1 - e^{-\frac{1}{2}\tau}\right)^{2/3}, \tag{50}$$

indicating its value for $\tau = 0$ by z_0, and for $\tau = \infty$ by z_∞.

The equation (25) may be written

$$v = v_0\frac{z_\infty - z}{z_\infty - z_0}, \tag{51}$$

where

$$v_0 = -\frac{3}{A^{1/3}}(z_\infty - z_0), \tag{52}$$

since for $\tau = 0$, $v = v_0$ and $z = z_0$.

From the definition (23) of A and (52) we obtain

$$v_0^4 = \tfrac{2}{3}\frac{h\nu_\alpha}{m_H c\mu}\frac{S_c w}{C_0}(z_\infty - z_0)^3. \tag{53}$$

The equation of continuity (18) with (51) gives

$$C = C_0\frac{z_\infty - z_0}{z_\infty - z}. \tag{54}$$

Substituting (54) in the ionisation formula (16), we obtain

$$C_a = \gamma C_0^2\frac{(z_\infty - z_0)^2}{(z_\infty - z)^2 e^{-\tau}}. \tag{55}$$

Substitution of (55) in the definition (41) of τ and integration gives, using (50),

$$(z_\infty - z_0)^2 a_c\gamma C_0^2\xi = z_\infty^2(1 - e^{-\tau}) - \tfrac{12}{5}z_\infty^{5/2}(z^{5/2} - z_0^{5/2}) + \tfrac{6}{7}z_\infty^{3/2}(z^{7/2} - z_0^{7/2}) + \tfrac{3}{2}z_\infty(z^4 - z_0^4) - \tfrac{3}{5}(z^5 - z_0^5). \tag{56}$$

For $\tau = \infty$, $\xi = h_0$, and hence

$$[(z_\infty - z_0)^2 a_c\gamma]C_0^2 h_0 = z_\infty^2 - \tfrac{12}{5}z_\infty^{5/2}(z_\infty^{5/2} - z_0^{5/2}) + \tfrac{6}{7}z_\infty^{3/2}(z_\infty^{7/2} - z_0^{7/2}) + \tfrac{3}{2}z_\infty(z_\infty^4 - z_0^4) - \tfrac{3}{5}(z_\infty^5 - z_0^5). \tag{57}$$

The distance to the plane of the nebula is given by $h = h_0 - \xi$, and hence

$$(z_\infty - z_0)^2 a_c\gamma C_0^2 h = z_\infty^2 e^{-\tau} - \tfrac{12}{5}z_\infty^{5/2}(z_\infty^{5/2} - z^{5/2}) + \tfrac{6}{7}z_\infty^{3/2}(z_\infty^{7/2} - z^{7/2}) + \tfrac{3}{2}z_\infty(z_\infty^4 - z^4) - \tfrac{3}{5}(z_\infty^5 - z^5). \tag{58}$$

whereas the ratio h/h_0 is obtained by dividing (58) by (57), and is a function of τ and ${\tau_0}'^3/A$ only.

For a star of given radius and temperature and given distance h_0 from the nebula, the S_c follows from (1) and the γ from (17) with $r = h_0$. For any value of the constant ${\tau_0}'^3/A$, the constants z_∞ and z_0 are known from (50). Then first C_0 is obtained from (57), and v_0 from (53). For a given τ, or z (50), h/h_0 follows then from (57) and (58), and v, C and C_a from (51), (54) and (55), so that these quantities are known as functions of the relative distance h/h_0 to the nebula. This space distribution being not readily accessible to observation, no examples of it have been given. We remark, however, that the distribution of all relative quantities v/v_0, C/C_0, C_a/C_{a0} as a function of the relative distance h/h_0 is completely determined by the one constant ${\tau_0}'^3/A$.

From the observational viewpoint, the luminosity distribution in v is of most interest. It can be obtained from (51), which is a direct consequence of the original formula (25). Substituting (32) in (51) we obtain, replacing z first by its definition (50),

$$\frac{v}{v_0} = \frac{\left(\frac{{\tau_0}'^3}{A} + 1\right)^{2/3} - \left(\frac{{\tau_0}'^3}{A} + 1 - L^{1/2}\right)^{2/3}}{\left(\frac{{\tau_0}'^3}{A} + 1\right)^{2/3} - \left(\frac{{\tau_0}'^3}{A}\right)^{2/3}}. \tag{59}$$

Differentiation with respect to L gives

$$\frac{dL}{dv/v_0} = 3\left[\left(\frac{{\tau_0'}^3}{A}+1\right)^{2/3} - \left(\frac{{\tau_0'}^3}{A}\right)^{2/3}\right] L^{1/2}\left(\frac{{\tau_0'}^3}{A}+1-L^{1/2}\right)^{1/3}. \qquad (60)$$

These two formulæ, (60) and (59), give the *shape of the spectral line* for the nebula viewed perpendicularly, expressed by means of the parameter $L^{1/2} = e^{-1/2\tau}$. The *width* of the spectral line follows from the value of v_0 obtained in the manner described. Examples of line profiles corresponding to ${\tau_0'}^3/A = 0{\cdot}311$ and $0{\cdot}0511$ are given in fig. 2 by the curves marked $1{\cdot}5$ and 1 (see p. 54).

To get the luminosity distribution in h, we first differentiate (50) with respect to $e^{-\tau}$, that is L (32),

$$\frac{dz}{dL} = -\tfrac{1}{3}e^{+\frac{1}{2}\tau}z^{-1/2}. \qquad (61)$$

Then differentiating (58) with respect to L, using (61) and (32), we obtain

$$[(z_\infty - z_0)^2 a_c \gamma {C_0}^2]^{-1}\frac{dL}{dh} = [{z_\infty}^2 + e^{+\frac{1}{2}\tau}(-2{z_\infty}^{5/2}z + {z_\infty}^{3/2}z^2 + 2z_\infty z^{5/2} - z^{7/2})]^{-1}. \qquad (62)$$

Equations (62) and (58) give the *luminosity distribution* dL/dh as a function of the *distance* h to the nebula, τ being the parameter. To introduce the relative distance h/h_0, one may multiply (62) by (57), at the same time dividing (58) by (57). This relative luminosity distribution is completely determined by the one constant ${\tau_0'}^3/A$.

We have seen in the foregoing that the solution, viz. the space distribution of all quantities concerned, is completely determined by the constants ${\tau_0'}^3/A$, h_0, γ, S_c and w. It was pointed out that the *relative* distributions, involving v/v_0, C/C_0 and h/h_0, are solely determined by the one constant ${\tau_0'}^3/A$.

Now the constant A by its definition (23) is a rather involved quantity, and for this reason it seems desirable to replace the ${\tau_0'}^3/A$ by some other constant allowing a simple physical interpretation. As such we take the constant $v_0\tau_0'$, which is the product of velocity and $d\tau/dv$ at $\tau = 0$, or the ratio of velocity to $dv/d\tau$ at this point.

Multiplying equation (52) by τ_0', and substituting the values z_∞ and z_0 from (50), we obtain the desired expression for $-v_0\tau_0'$ in terms of ${\tau_0'}^3/A$

$$-v_0\tau_0' = 3\left[\left(\frac{{\tau_0'}^3}{A}\right)^{1/3}\left(\frac{{\tau_0'}^3}{A}+1\right)^{2/3} - \frac{{\tau_0'}^3}{A}\right], \qquad (63)$$

from which the following table is obtained :—

TABLE I

$-v_0\tau_0'$	2	1·9	1·75	1·5	1·25	1·0	0·75	0·5	0·25	0
${\tau_0'}^3/A$	$+\infty$	2·98	0·92	0·311	0·126	0·0510	0·0187	0·0050	0·00060	0

Intermediate values may be obtained by plotting $\log({\tau_0'}^3/A)$ against the $-v_0\tau_0'$ of this table.

Values of $-v_0\tau_0'$ between 2 and $+\infty$ do not occur, whereas negative values of this quantity would mean that the $dv/d\tau$ has at least once reversed its sign, and then entirely different portions of the nebula may interact, which is precluded by our assumptions.

For $-v_0\tau_0' = 1{\cdot}5$ and 1, the *shape of the spectral line* calculated by means of Table I is given in fig. 2 by the curves marked 1·5 and 1. (Like for the two other curves, the left-hand portion is uncertain, but it has not been dotted.) We remark that each curve intersects the vertical axis $v/v_0 = 1$ at an ordinate equal to $-v_0\tau_0'$, as may also be seen from equation (46). Hence the intensity in the spectral line at $v = v_0$ is finite, except in the case $A >> \tau_0'^3$. The identification of spectral line intensity at $v = v_0$ with the value $-v_0\tau_0'$ is a convenient guide for judging the shape for other values than those given in the figure.

The previous treatment gives no clue as to the value of $-v_0\tau_0'$, which is simply a constant occurring in the boundary conditions at $\tau = 0$. From a physical point of view this is unsatisfactory, since it seems that the $-v_0\tau_0'$ should be determined by the physical situation : either by boundary conditions far away from the nebula, or by the way in which the stationary state had been set up when the star approached the nebula. There remains here a problem still to be solved.

6. *Application to the Orion Nebula.*—The observational material for the Orion Nebula dealt with in this section consists of Trumpler's * parallax, the photometry and resulting dimensions by Hopmann † and the investigation on line width by Baade, Goos, Koch and Minkowski.‡

According to Hopmann, the exciting stars of the great nebula in Orion are the two or three brightest stars of the trapezium (θ_1 Orionis) and the brightest star of θ_2 Orionis, the spectral type of which is now known to be O9k (Plaskett and Pearce, *Victoria Publ.* V, No. 2); the total photographic magnitude is $4^m{\cdot}3$. For the total visual magnitude of the nebula Hopmann finds $3^m{\cdot}3$, so that $m_{*\mathrm{ph}} - m_{\mathrm{n\,vis}} = 1^m{\cdot}0$. From the table given by the author (*Zeitschr. f. Astrophys.*, **2**, 1, 1931, Table VI) then follows a temperature of 28,000° obtained by the nebulium method (the value 23,000° Hopmann gives is due to an oversight on his part). Allowing a factor 2 for the fact that the nebula does not completely surround the star, one would obtain 32,000°. On the other hand, not all of the light of the nebula is due to O_{III}, so that the round figure 30,000° for the average temperature of the exciting stars will be adopted.

We shall make use of the simplified picture adopted by Hopmann, in which the nebula is approximated by a plane and the stars are replaced by one situated at their luminous centre. Hopmann gave also a more refined treatment, but his simplification is best suited to our needs. The isophotes are then circles, seen on the photograph as ellipses, and from the ratio of major to minor axis and other evidence Hopmann finds that the plane is

* R. J. Trumpler, *Publ. Astr. Soc. Pac.*, **43**, 255, 1931.

† J. Hopmann, *Veroff. d. Univ. Sternw. Leipzig*, **42**, Heft 4, 1934.

‡ W. Baade, F. Goos, P. P. Koch and R. Minkowski, *Zeitschr. f. Astrophys.*, **6**, 355, 1933.

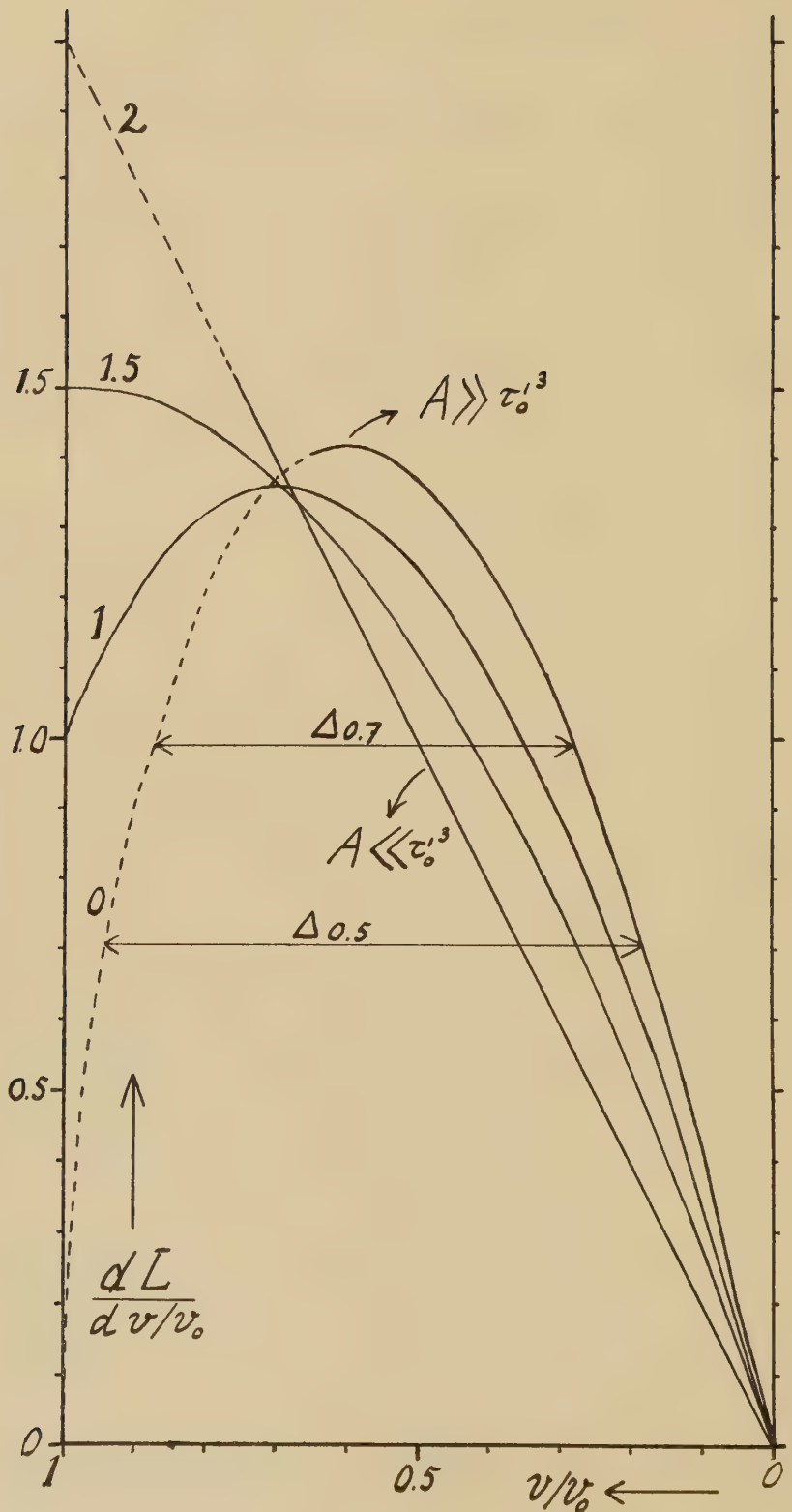

FIG. 2.—*Theoretical shapes of spectral lines for a diffuse nebula viewed perpendicularly. The number to the left and above each curve represents the value of* $-v_0\tau_0'$.

tilted over an angle θ (our notation) of about 45° with respect to the celestial sphere. It is clear that the distance over which the luminosity on the photograph varies appreciably is also the order of magnitude of the distance of the resultant star to the nebula. Hopmann finds for the distance of star to nebula the value $63{\cdot}7''\sqrt{\sec\theta}$, hence $75{\cdot}8''$ for $\theta = 45°$.* With Trumpler's value of 540 parsecs for the distance, we have

$$h_0 = 40{,}000 \text{ astr. units} = 6{\cdot}0 \times 10^{17} \text{ cm.}$$

According to Trumpler, stars involved in the Orion Nebula have been weakened by about $0^{m}{\cdot}6$ on account of absorption, so that the actual brightness of the resulting star is $3^{m}{\cdot}7$, and its absolute photographic brightness $-5^{m}{\cdot}0$. For $T = 30{,}000°$ this corresponds to an absolute bolometric brightness $M_{*\text{bol}} = -7^{m}{\cdot}1$ (*Zeitschr. f. Astrophys.*, **2**, 329, 1931, Table II). From the Stefan-Boltzmann Law one then has (*ibid.*)

$$\log R/R_{\odot} = +8{\cdot}5 - 2\log T - \tfrac{1}{5}M_{*\text{bol}} = +1{\cdot}0,$$

so that the radius of the resulting star is 10 times that of the Sun or

$$R = 6{\cdot}95 \times 10^{11} \text{ cm.}$$

We now proceed with the comparison of the theoretical predictions with the observed shape and width of the spectral line H_{β}. Detailed observations of line profiles have not been made by Baade and his co-workers, the main reason being probably that the interferometer pattern shows some irregularities interpretable as change of average velocity from point to point. Nevertheless the registration curves of their fig. 9 seem to show that this profile is fairly symmetrical, so that we are probably close to the case $-v_0\tau_0' = 0$ of our fig. 2. Perhaps the case $-v_0\tau_0' = 1$ would still be permissible, but a value higher than this appears very unlikely.

Concluding thus from the symmetry that we have the case $A >> \tau_0'^3$, we obtain from (1), (17) and (44), with the above values of T, $r = h_0$ and R,

$$S_e = 1{\cdot}58 \times 10^{11} \text{ quanta per cm.}^2 \text{ per sec.}, \qquad C_0 = 1120 \text{ atoms per cm.}^3.$$

In section 3 we gave the estimate that the ratio of thermal width to observed width is of the order 0·1, hence we may assume $w/v_0 = 0{\cdot}1$. Substituting S_e, C_0 and w/v_0 in (38) (where the universal constants and μ (3*a*) have been substituted and both sides have been divided by v_0), we get

$$v_0^3 = 2{\cdot}67 \times 10^{12}\frac{S_e}{C_0}\frac{w}{v_0} = 38{\cdot}7 \times 10^{18} \text{ (cm./sec.)}^3. \tag{64}$$

Hence we have $v_0 = 34$ km./sec.

Now the C_0 following for a given C_a from the ionisation formula (28) is certainly much too high, because the formula supposes only captures

* We remark that Hopmann assumes a surface luminosity indirectly proportional to the square of the distance from the star, but this should be multiplied by the secant of the angle of this radius vector with the normal to the plane. Nevertheless, his determination must be considered approximately correct and certainly gives the order of magnitude, which mainly interests us at present.

on the first level, and because it assumes that no energy is lost in electron excitation of the nebulium lines. It seems that it might well be 10 times too high, so that one should take γ 100 times higher, or finally in (44) the C_0 should come out 10 times lower. Doing this, we get $v_0 = 34\sqrt[3]{10} = 73$ km./sec. The width at which the intensity has gone down to one-half of its maximum value is, according to fig. 2, $\Delta_{0\cdot5} = 0\cdot760\ v_0$, similarly $\Delta_{0\cdot7} = 0\cdot592\ v_0$ for the nebula viewed perpendicularly. This should be multiplied by $\cos 45° = 0\cdot708$, since the plane is tilted over 45°, and so we get

$$\Delta_{0\cdot7} = 14 - 30 \text{ km./sec.},$$

the higher value referring to the ionisation of 0·1 of that given by the formula. The observed $\Delta_{0\cdot7}$ for the region near the centre of light is about 0·4 A. for H_β (Baade, *loc. cit.*, fig. 10) or 25 km./sec.

The predicted line width of the theory is therefore in good agreement with the observed value. For the case $-v_0\tau_0' = 1$ formula (53) would give a value of v_0 about 0·6 times the above value, whereas $\Delta_{0\cdot7} = 0\cdot67\ v_0$ (fig. 2), so that $\Delta_{0\cdot7} = 10 - 21$ km./sec. In any case there seems to be agreement as regards order of magnitude.

The foregoing presents evidence that the widening of lines in the Orion Nebula is due to the force of radiation pressure of L_α. If this force kept the same value as in the stationary case it would be catastrophic, but it is the expansion itself which cuts the force down. This idea was set forth in our previous paper I. In the present paper the force and the velocity it imparts have actually been calculated and shown to be in agreement with observation for a fairly symmetrical line contour.

It is intended to apply the dynamical methods developed in the present paper to the dynamics of Wolf-Rayet stars and planetary nebulæ.

The author wishes to express his gratitude to Professor H. H. Plaskett for fruitful discussions and suggestions and for the facilities extended to him.

University Observatory,
Oxford :
1936 *August.*

THE STELLAR PROBLEM OF THREE BODIES: I. APPLICATION OF SATELLITE THEORY.

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1. In this and two or three following papers I propose to take up the stellar problem of three bodies, chiefly from the point of view of the observer, by limiting the solution to those terms which may be capable of being detected by observation at the present time and in the immediate future. In this first paper it will be shown how a literal satellite theory like that of Delaunay can, under certain conditions, be adapted to the stellar problem. The principal defect of this method is the fact that the results are useful only when the eccentricities and the inclination are small. In the second

paper this defect is remedied. The general equations are set up and a solution which is probably sufficiently accurate for almost any of the known problems is obtained. In the third paper the general problem is treated and, in particular, it is shown how the motions of the apse and node can be obtained. In a fourth paper I hope to apply the theory to the system ξ Ursæ Majoris and perhaps to one or two other interesting systems. I shall also apply it to satellite theory and show how certain portions can be carried without extensive calculation to an accuracy higher even than that reached by Delaunay in his lunar theory. An interesting feature of the method is the nearly complete avoidance of the use of infinite series, so that the expressions obtained are available for almost any values of the eccentricities and inclination.

2. The property common to the satellite and stellar problems and that which determines the method to be used for approximation to the solution is the configuration in which two of the bodies (planet and satellite, or two stars) form a close pair, while the third body (Sun or another planet, or a third star) is at a distance from them which is large compared with their distance from one another. The two problems can be given the same mathematical form, the only difference being a factor of the disturbing function which is less than unity in the stellar problem and which is taken to be equal to unity in the satellite problem. Owing to certain peculiarities of the problem some care is necessary in order to find out how the factor enters into the perturbations.

3. The assumption made throughout is that *no periodic term which has the squares or products of the ratio of the periods and that of the mean distances as a factor of its coefficient can be detected by observation.* As for the mean motions of the apse and node which have the square of the ratio of the periods as a factor, it may be possible to detect them after the lapse of a sufficiently long interval of observation. Consequently, in the third paper those portions which have the square and cube of this ratio as a factor are calculated. The latter is included because of the presence of large numerical coefficients : in the case of the Moon the two portions have nearly the same numerical value. Interesting and unexpected results appeared in this part of the work. One of them is the deduction of the terms dependent on the eccentricity e' of the orbit of the third body from those independent of this parameter, by means of a simple factor, which is $(1-e'^2)^{-\frac{3}{2}}$ for the terms factored by m^2 and is $(1+\frac{2}{3}e'^2)(1-e'^2)^{-3}$ for the terms factored by m^3. Most of the results are obtained without the use of infinite series, and where such series have to be used the convergence is rapid.

A certain amount of known satellite theory must be given here in order to make the argument clear. For the analytical developments leading to the principal formulæ the reader is referred to standard treatises.

4. As usual, the motion of the distant star is referred to the centre of mass of the close pair as origin, and that of one member of the pair to the other as origin. This has the effect of making the perturbations of the outer star by the close pair a minimum : these have, in fact, the square of the ratio of the distances as a factor and are negligible in both satellite

and stellar problems. The plane of the orbit of m_3 will be chosen as plane of reference.

For the motion of the close pair—that of m_2 referred to m_1 and disturbed by the outer body m_3—the disturbing function is *

$$m_3\frac{r^2}{r'^3}\left\{\tfrac{3}{2}\sigma^2-\tfrac{1}{2}+\frac{m_1-m_2}{m_1+m_2}\cdot\frac{r}{r'}(\tfrac{5}{2}\sigma^3-\tfrac{3}{2}\sigma)+\cdots\right\},$$

where σ is the cosine of the angle between the two radii r, r'. The adopted limitation confines the function to the two terms given.

The method for obtaining the results will be a modification of that of Delaunay developed in *Planetary Theory* by Brown and Shook. In summary, this is as follows :—

The set of canonical elements to be used is obtained by replacing, in the elliptic expressions for the co-ordinates and velocities, the ordinary elements by

$$\begin{aligned} w_1 &= nt+\epsilon = \text{mean long.}, & c_1 &= \sqrt{\mu a} = na^2,\\ w_2 &= \text{long. of apse}, & c_2 &= c_1\sqrt{1-e^2}-c_1,\\ w_3 &= \text{long. of node}, & c_3 &= (\cos i-1)(c_1+c_2), \end{aligned}$$

where

$$\mu = (m_1+m_2),$$

and the equations of motion are :

$$\frac{dc_i}{dt}=\frac{\partial R}{\partial w_i}, \qquad \frac{dw_i}{dt}=-\frac{\partial}{\partial c_i}\left(\frac{\mu^2}{2c_1^2}+R\right), \qquad i=1,\ 2,\ 3.$$

The equations for $i=1$ with $n=\mu^2/c_1^3$ may be written

$$\frac{dc_1}{dt}=\frac{\partial R}{\partial w_1}, \qquad \frac{dw_1}{dt}=n-\frac{\partial R}{\partial c_1}.$$

The perturbations of the first order are given by

$$\frac{d}{dt}\delta n=-3\frac{n}{c_1}\frac{\partial R}{\partial w_1}, \qquad \frac{d}{dt}\delta w_1=\delta n-\frac{\partial R}{\partial c_1},$$

so that a double integration is needed except for those portions of R which are independent of w_1.

The function R is expressed as a sum of harmonic terms with coefficients depending on the c_i and angles on w_i, $w_i'=n't+\epsilon'$, $w_2'=\varpi'$, the latter pair arising from the elliptic motion of m_3.

5. The solution really proceeds by approximation, the method of procedure being concealed by a definite change of variables. There is a large range of possibilities in making the change. That adopted here is the following : Put

$$\frac{\mu^2}{2c_1^2}+R=R_c+R_p=R_c+\Sigma\bar{K}\cos(iw_1+i'w_1'+\bar{W}),$$

* Brown, *Lunar Theory*, chap. i.

where R_p contains those terms for which i, i' are not both zero, and $\bar{W}$ is a function of w_2, w_3, w_2'.

Define S by

$$S = \sum \frac{\bar{K}}{in + i'n'} \sin (iw_{10} + i'w_1' + \bar{W}_0),$$

where the symbols w_{i0} replace the w_i in the angle and are defined by

$$w_i = w_{i0} - \frac{\partial S}{\partial c_i}.$$

Three more new variables c_{i0} are defined by

$$c_{i0} = c_i - \frac{\partial S}{\partial w_{i0}},$$

and the equations are transformed from the variables c_i, w_i to c_{i0}, w_{i0}.

When this is done, the equations for the new variables are found to be *

$$\frac{dc_{i0}}{dt} = \frac{\partial R_2}{\partial w_{i0}}, \qquad \frac{dw_{i0}}{dt} = -\frac{\partial R_2}{\partial c_{i0}},$$

where

$$R_2 = R_c + R_p - R_{p0} - \frac{\mu^2}{2c_1^2} + \frac{\mu^2}{2c_{10}^2} - \frac{3}{2}\frac{\mu^2}{c_{10}^4}(c_1 - c_{10})^2,$$

R_{p0} being what R_p becomes when we replace the letters w_i by w_{i0} (but not c_i by c_{i0}). Finally R_2 is expressed in terms of the new variables.

If R_p contains a constant factor α, it is evident from their mode of formation that $c_i - c_{i0}$, $w_i - w_{i0}$ contain the same factor and that, if we expand in powers of α, $R_p - R_{p0}$ will have the factor α^2. Thus, the change of variables causes the replacement of R_p which has the factor α by terms which have the factor α^2. Also $\mu^2/2c_1^2$ is replaced by $\mu^2/2c_{10}^2$ plus terms which have the factor α^2.

If terms having the factor α^2 be neglected, we can put $c_i = c_{i0}$ in S, and the solution of the equations is reduced to that of the equations

$$\frac{dc_{i0}}{dt} = \frac{\partial R_c}{\partial w_{i0}}, \qquad \frac{dw_{i0}}{dt} = -\frac{\partial R_c}{\partial c_{i0}}, \qquad i = 2, 3$$

$$c_{10} = \text{const.} \qquad \frac{dw_{10}}{dt} = \text{const.} - \frac{\partial R_c}{\partial c_{10}},$$

with the integral $R_c = \text{const.}$

6. The principal factor of the disturbing function is

$$m_3 \frac{r^2}{r'^2} = \frac{m_3}{m_1 + m_2 + m_3} \cdot \frac{n'^2}{n^2} \cdot \left(\frac{r}{a}\right)^2 \left(\frac{a'}{r'}\right)^3 \cdot \frac{\mu}{a'},$$

where

$$\mu = m_1 + m_2 = n^2 a^3, \qquad m_1 + m_2 + m_3 = n'^2 a'^3.$$

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* Brown and Shook, chap. vi.

The small factor α, mentioned above, is therefore λm^2, where

$$m = \frac{n'}{n}, \qquad \lambda = \frac{m_3}{m_1 + m_2 + m_3}.$$

The factor λ is put equal to 1 in satellite theory, and our chief concern here is to find out where it is to be inserted in satellite theory to obtain the stellar theory.

7. The terms in the elliptic expansion of R will be divided into three classes :

(1) The terms of R_p in which $i \neq 0$ (§ 5). These are called *short period* and will be denoted by R_s.

(2) The terms in which $i = 0$, $i' \neq 0$. These are called *long period* and will be denoted by R_l.

(3) The terms in which $i = 0$, $i' = 0$, but which contain two * or more of the arguments w_2, w_3, w_2'. These will be called the *apse-node* terms. They are included with the terms independent of w_i, w_i' in R_c.

It is assumed that the terms for which $in + i'n' (i, i' \neq 0)$ is small have such small coefficients that they lie wholly outside the range of observation.

Consider first the short-period terms. These have all finite divisors in S and consequently in the perturbations of the elements. Hence in the first approximation to the elements all such terms have the factor λm^2.

The long-period terms are similar but have the divisor n'. Since w_1 is absent, $\delta n = 0$ and the first power of the divisor only arises. Hence the long-period terms in the elements have the factor λm.

The mean motions of the apse and node are obtained from the terms in R_c independent of w_i, w_i', and have the factor λm^2.

These motions must be inserted in the arguments of the apse-node terms. Before integration these terms have the factor λm^2, so that after the integration this factor disappears.

The rule for deducing the results for the stellar problem from those for the satellite problem is therefore as follows :—

Confine the coefficient of each periodic term to the lowest power of m present in that coefficient, neglecting all terms in which this lowest power is greater than the second. Multiply all coefficients in which this lowest power is m or m^2 by λ and leave unchanged those in which it is m^0. The latter part of the rule obviously includes the elliptic terms in the co-ordinates.

8. We have still to find out whether any change in this rule is produced by the second approximation—that is, by the use of R_2 in § 5.

All the second-order terms in R_2 arise as the product of two cosines or of two sines. When these terms are put in the form of the sum or difference of two cosines, we have R_2 expressed as a sum of harmonic terms which will contain short, long and apse-node terms. If the argument of one cosine is short and that of the other long or apse-node, the resulting terms in R_2 will be short. In other cases we shall get short, long and apse-node terms.

* All arguments are differences of the given angles.

First, the short-period terms will produce terms with the factor $\lambda^2 m^4$ in R_2. Any long-period terms which arise will therefore have the factor $\lambda^2 m^3$ after integration and any apse-node terms the factor λm^2.

Second, the long-period terms will produce terms with the factor $\lambda^2 m^3$ in R_2, and such terms will produce terms with the factors $\lambda^2 m^2$, λm in the long-period and apse-node terms in the elements.

Finally, the apse-node terms will produce terms with the factor λm^2 in R_2, and consequently terms independent of this factor in the elements.

The rule is therefore still good for the second approximation and obviously for all higher approximations.

9. In the second paper of this series it will be seen that there is only one apse-node term in the elliptic expansion of the first term of R, and that this has the argument $2\varpi - 2\theta$. As the principal terms in the mean motions of ϖ, θ are

$$\tfrac{3}{4}\lambda m^2 n, \qquad -\tfrac{3}{4}\lambda m^2 n,$$

the motion of this argument is $3\lambda m^2 n$.

An addition to the rule can be made for the terms in the mean motions of the apse and node which have the factor m^3, namely, that these portions have the factor $\lambda^2 m^3$ in the stellar problem.

The second term of R contains the additional factor

$$\frac{m_1 - m_2}{m_1 + m_2} \cdot \frac{a}{a'}.$$

The limitation of § 3 will therefore require us to consider only the apse-node terms in this part.

There is one possible term which might constitute an exception to the rule, namely, that with argument $2\varpi + 2\theta$, the principal part of whose motion will have the factor λm^3. It is not certain that this argument can be present in the elliptic expansion of R. If it is present, however, I show in the second paper that it must have the factor $\lambda m^2 (a/a')^3$—that is, λm^4 in R, and consequently the factor m after integration. In any case, however, the term is negligible as it has the additional factor $e^2 e'^4 \sin^2 i$.

The rule was proved by means of the consideration of the long, short and apse-node terms in the elements, but its final form is independent of the arguments. Thus the rule is equally valid when the disturbed values of the elements are substituted in the elliptic expressions for the co-ordinates. This statement is necessary because long-period terms in the elements give rise to both long- and short-period terms in the co-ordinates, and similarly for terms with other periods.

Yale Observatory :
1936 *August.*

THE STELLAR PROBLEM OF THREE BODIES: II. THE EQUATIONS OF MOTION WITH A SIMPLIFIED SOLUTION.

Ernest W. Brown, Sc.D., F.R.S.

1. In the first paper it was shown how a literal satellite theory could be adapted to the stellar problem. The use of this theory is limited to small eccentricities and inclination. In this second paper I give a solution which is valid for almost all values of the eccentricities and inclination under the adopted limitation which consists of the neglect of all periodic terms whose coefficients contain the squares or product of the ratio of the periods and distances as factors.

The most important of these parameters from the observational point of view is the eccentricity e' of the outer orbit. The only apparent limitation on its value is that which arises in the expansion of the disturbing function in powers of r/r'. In considerations of convergence it appears that this should be replaced by

$$\frac{1}{1-e'^2}\cdot\frac{a}{a'}.$$

It is evident that e' may have a large value without seriously affecting this ratio from the point of view of convergence.

No limitation on the eccentricity of the close pair or on the mutual inclination of the orbits appears. The simplified solution given below will probably satisfy all present needs of observers. The simplification consists mainly in the neglect of the mean motions of the apse and node, for the observational determination of which it is doubtful whether a sufficient interval has yet elapsed with any system in which the adopted limitation is valid.

With respect to this fundamental limitation it may perhaps be useful to point out the fact that in all perturbations of the elements of the first order with respect to the disturbing force, the product of the coefficient of any term by the motion of its argument contains the factor λm^2. The three cases are :

short period,	coefficient factor	λm^2,	argument factor	1 ;
long	,,	,,	λm,	,,	,, m ;
apse-node,	,,	,,	1,	,,	,, λm^2,

the second of these being the case treated here.

2. With the same notation as before, the disturbing function is

$$R = m_3\frac{r^2}{r'^3}\left\{\tfrac{3}{2}\sigma^2 - \tfrac{1}{2} + \frac{m_1-m_2}{m_1+m_2}\frac{r}{r'}\left(\tfrac{5}{2}\sigma^3 - \tfrac{3}{2}\sigma\right)\right\}.$$

Let v, v' be the longitudes of m_2, m_3 reckoned in the usual manner and θ, i the longitude of the node and inclination of the orbit of m_2 on that of m_3. Then

$$\begin{aligned}\sigma &= \cos(v-\theta)\cos(v'-\theta) + \cos i \sin(v-\theta)\sin(v'-\theta)\\ &= \tfrac{1}{2}(1+q)\cos(v-v') + \tfrac{1}{2}(1-q)\cos(v+v'-2\theta),\end{aligned}$$

where

$$q = \cos i,$$

whence

$$\sigma^2 = \tfrac{1}{4}(1+q^2) + \tfrac{1}{8}(1+q)^2 \cos(2v - 2v') + \tfrac{1}{8}(1-q)^2 \cos(2v + 2v' - 4\theta) + \tfrac{1}{4}(1-q^2)\cos(2v - 2\theta) + \tfrac{1}{4}(1-q^2)\cos(2v' - 2\theta),$$

$$\sigma^3 = \tfrac{3}{32}[\{(1+q)^3 + 2(1-q)(1-q^2)\}\cos(v - v') + \{(1-q)^3 + 2(1+q)(1-q^2)\}\cos(v + v' - 2\theta) + (1+q)(1-q^2)\cos(3v - v' - 2\theta) + (1-q)(1-q^2)\cos(3v + v' - 4\theta)\}]$$

+terms containing $3v'$.

3. Let f, ϖ be the true anomaly and longitude of the apse of the close pair, so that $v = f + \varpi$. If l be the mean anomaly, f, v are functions of l, ϖ, e. In the first paper we have seen that the fundamental limitation involves the neglect of all short-period terms in R; that is, of all terms whose arguments contain l. Hence we need only those portions of r^2, $r^2\cos 2f$, $r^3\cos f$, $r^3\cos 3f$, expanded into Fourier series with argument l, which are independent of l.

These are easily found by the use of X, the eccentric anomaly. We have

$$r\cos f = a\cos X - ae, \qquad r = a - ae\cos X, \qquad dX/dl = a/r,$$

and X, l take the values 0, 2π together. Hence the term independent of l in the expansion of $r^i\cos jf$ is

$$\frac{1}{2\pi}\int_0^{2\pi} r^i \cos jf\, dl = \frac{1}{2\pi a}\int_0^{2\pi} r^{i+1}\cos jf\, dX.$$

Hence, by expressing the integrand in terms of X, we find that the non-periodic terms in

$$r^2, \qquad r^2\cos 2f, \qquad r^3\cos f, \qquad r^3\cos 3f$$

are

$$(1+\tfrac{3}{2}e^2)a^2, \qquad \tfrac{5}{2}e^2a^2, \qquad (-\tfrac{5}{2}e - \tfrac{15}{8}e^3)a^3, \qquad -\tfrac{35}{8}e^3a^3.$$

4. If we substitute elliptic values for the co-ordinates in R, the time is now present explicitly only through r', f', or, if we put

$$\frac{1}{r'} = \frac{1 + e'\cos f'}{a'(1-e'^2)},$$

only through f'.

Let us therefore change the independent variable from t to f′ in the equations of motion.

It is mainly due to this change that the problem and its solution can be so greatly simplified. It is also responsible for the discovery of certain new properties of the mean motions of the apse and node to be developed in Paper III.

Since

$$r'^2\frac{df'}{dt} = n'a'^2(1-e'^2)^{\frac{1}{2}}$$

the equation $da/dt = \partial R/\partial\beta$ becomes

$$\frac{da}{df'} = \frac{\partial}{\partial\beta}\left(\frac{r'^2}{n'a'^2}(1-e'^2)^{-\frac{1}{2}}R\right) \equiv \frac{\partial R'}{\partial\beta}.$$

5. With the use of the first term of R, the corresponding term of R' can be written

$$R' = k_1(1+e'\cos f')[(1+\tfrac{3}{2}e^2)\{-\tfrac{1}{3}+q^2+(1-q^2)\cos(2f'+2\varpi'-2\theta)\}$$
$$+\tfrac{5}{2}e^2\{(1-q^2)\cos(2\varpi-2\theta)+\tfrac{1}{2}(1+q)^2\cos(2f'+2\varpi'-2\varpi)$$
$$+\tfrac{1}{2}(1-q)^2\cos(2f'+2\varpi'+2\varpi-4\theta)\}],$$

where

$$k_1 = \tfrac{3}{8}m_3\frac{a^2}{a'^3n'}(1-e'^2)^{-\frac{3}{2}}.$$

In the second term of R' arising from the second term of R, it will presently be shown that we need only the terms independent of f'. The functions of f' which may have constant parts and which appear in this second term are

$$(1+e'\cos f')^2\cos f', \qquad (1+e'\cos f')^2\cos 3f'.$$

These constant parts are $\frac{1}{2}e'$, 0. The second term of R' is therefore

$$k_1k_2(\tfrac{5}{16}e+\tfrac{15}{64}e^3)\{(1+11q-5q^2-15q^3)\cos(\varpi-\varpi')$$
$$+(1-11q-5q^2+15q^3)\cos(\varpi+\varpi'-2\theta)\}$$
$$-k_1k_2\cdot\tfrac{175}{64}e^3\{(1+q-q^2-q^3)\cos(3\varpi-\varpi'-2\theta)$$
$$+(1-q-q^2+q^3)\cos(3\varpi+\varpi'-4\theta)\},$$

where

$$k_3 = \frac{e'}{1-e'^2}\cdot\frac{a}{a'}\cdot\frac{m_1-m_2}{m_1+m_2}.$$

The first term of R' with

$$a=\frac{L^2}{\mu}, \qquad e^2=1-G^2/L^2, \qquad q=H/G, \qquad \varpi=g+h, \qquad \theta=h,$$

and with L, n constant, is now inserted in the transformed equations of Delaunay,

$$\frac{dG}{df'}=\frac{\partial R'}{\partial g}, \qquad \frac{dH}{df'}=\frac{\partial R'}{\partial h}, \qquad \frac{dg}{df'}=-\frac{\partial R'}{\partial G}, \qquad \frac{dh}{df'}=-\frac{\partial R'}{\partial H},$$

$$L\frac{d}{df'}(l+g+h-nt-\epsilon) = -L\left(\frac{\partial R'}{\partial L}+\frac{\partial R'}{\partial G}+\frac{\partial R'}{\partial H}\right)$$
$$= -4R'-(L-G)\frac{\partial R'}{\partial G}-(L-H)\frac{\partial R'}{\partial H},$$

since R' is a homogeneous function of L, G, H, of the fourth degree.

6. For the solution of these equations, to all the elements in the right members are given constant values. The only variable present is f' in the form of cosines and sines of its first three multiples. The integration is therefore simple and exact. For brevity of expression, put

$$C(\alpha)=\cos(2f'+\alpha)+e'\cos(f'+\alpha)+\tfrac{1}{3}e'\cos(3f'+\alpha),$$
$$S(\alpha)=\sin(2f'+\alpha)+e'\sin(f'+\alpha)+\tfrac{1}{3}e'\sin(3f'+\alpha),$$

these being the integrals of

$$-2(1+e'\cos f')\sin(2f'+\alpha), \qquad 2(1+e'\cos f')\cos(2f'+\alpha).$$

After the formation of the equations we can return to the more familiar variables e^2, $q = \cos i$. The results are :

$$\delta G = -5k_1e'e^2(1-q^2)\sin f' \sin(2\varpi - 2\theta) - \tfrac{5}{4}k_1e^2(1+q)^2C(2\varpi' - 2\varpi) + \tfrac{5}{4}k_1e^2(1-q)^2C(2\varpi' + 2\varpi - 4\theta),$$

$$\delta H = -k_1(1+\tfrac{3}{2}e^2)(1-q^2)C(2\varpi' - 2\theta) - \tfrac{5}{4}k_1e^2(1+q)^2C(2\varpi' - 2\omega) - \tfrac{5}{4}k_1e^2(1-q)^2C(2\varpi' + 2\varpi - 4\theta),$$

$$\begin{aligned} G\delta g = {} & k_1e'\{5q^2 - 1 + e^2 + 5(1-e^2-q^2)\cos(2\varpi' - 2\theta)\}\sin f' \\ & + k_1(\tfrac{3}{2} - \tfrac{3}{2}e^2 - \tfrac{5}{2}q^2)S(2\varpi' - 2\theta) \\ & + \tfrac{5}{4}k_1(1+q)(1+q-e^2)S(2\varpi' - 2\varpi) \\ & - \tfrac{5}{4}k_1(1-q)(1-q-e^2)S(2\varpi' + 2\varpi - 4\theta), \end{aligned}$$

$$\begin{aligned} H\delta h = {} & -k_1e'q^2\{2 + 3e^2 + 5q^2e^2\cos(2\varpi - 2\theta)\sin f'\} \\ & + k_1q^2(1+\tfrac{3}{2}e^2)S(2\varpi' - 2\theta) \\ & + \tfrac{5}{4}k_1qe^2(1+q)S(2\varpi' - 2\varpi) - \tfrac{5}{4}k_1qe^2(1-q)S(2\varpi' + 2\varpi - 4\theta), \end{aligned}$$

$$\begin{aligned} L\delta(nt+\epsilon) = {} & -4k_1e'\{(-\tfrac{1}{3} + q^2)(1+\tfrac{3}{2}e^2) + \tfrac{5}{2}e^2(1-q^2)\cos(2\varpi - 2\theta)\}\sin f' \\ & - 2k_1(1-\tfrac{3}{2}e^2)(1-q^2)S(2\varpi' - 2\theta) \\ & - \tfrac{5}{2}k_1e^2\{(1+q)^2S(2\varpi' - 2\varpi) + (1-q)^2S(2\varpi' + 2\varpi - 4\theta)\} \\ & + (L-G)\delta g + (L-H)\delta h, \end{aligned}$$

with

$$k_1 = \frac{3}{8}\frac{m_3}{m_1 + m_2 + m^3}\frac{n'}{n}L.$$

The formula for the perturbation of the mean longitude requires some explanation. Since R' enters into the equations of motion only through its derivatives, it contains an additive arbitrary constant. Let the whole constant part of the right-hand member be C. Then

$$\begin{aligned} \delta(nt+\epsilon) &= Cf' + \text{periodic perturbations} \\ &= C(n't + \epsilon' - \varpi' + \text{eq. of centre}) + \text{periodic perturbations}. \end{aligned}$$

But as n is defined as the coefficient of t in the longitude we have $Cn' = 0$ and this makes $Cf' = 0$. Thus the term factored by f' in $\delta(nt+\epsilon)$ is put equal to zero, and this includes such terms in δg, δh in the last line, and, as we are neglecting the mean motions of the apse and node, also in the values of δg, δh as set down above.

The variations of e, q, ϖ, θ are obtained from the above expressions by the formulæ

$$L\delta e = -\frac{\sqrt{1-e^2}}{e}\delta G, \qquad L\delta q = -L\sin i\delta i = \frac{\delta H - q\delta G}{\sqrt{1-e^2}},$$

$$\delta\varpi = \delta g + \delta h, \qquad \delta\theta = \delta h.$$

The constant L disappears.

7. The calculations are carried out by using one of the usual tables giving the true anomly f' in terms of the mean anomaly l' for each date. The values of $\cos f'$, $\sin f'$, $C(\alpha)$, $S(\alpha)$ are then found and inserted in the results for the elements. The final results are obtained by inserting these values of the elements in the elliptic expressions for the co-ordinates.

A number of the terms will usually be insensible. Hence it is best to calculate first the coefficients in $\delta G/e$, δH, $e\delta g$, $i\delta h$, in order to find out those which have to be retained. It may be found then that some of the functions $C(\alpha)$, $S(\alpha)$ are not needed.

8. *Case of e small.*—In many stellar systems the eccentricity of the close pair is very small. In such cases ek_1 will not be observable. We then have

$$\delta e = 0, \qquad e\delta\varpi = 0,$$

$$L\delta q = -k_1(1-q^2)C(2\varpi' - 2\theta),$$

$$H\delta\theta = -2k_1e'q^2 \sin f' + k_1q^2S(2\varpi' - 2\theta),$$

$$L\delta(nt+\epsilon) = -4k_1e'(-\tfrac{1}{3}+q^2)\sin f' - 2k_1(1-q^2)S(2\varpi' - 2\theta) + (L-H)\delta\theta,$$

where $$H = L\cos i = Lq.$$

Thus the only sensible perturbations may be those of the plane of the orbit and those in the mean longitude.

Yale Observatory, 1936 *October*.

Note on Errors in Fundamental Equations in "Planetary Theory," by Ernest W. Brown and C. A. Shook (Camb. Univ. Press, 1933).—As these errors are likely to give trouble if the uncorrected equations are used, a note here appears to be advisable. The principal errors are :

P. 33. Equation (11) should read:

$$F_2 = \frac{F_1/\mu' - \frac{1}{2}\rho^2 e' \cos(v' - \varpi')}{1 + e'\cos(v' - \varpi')},$$

i.e. "$F_1/\mu' -$" *for* "$F_1 +$."

P. 136. The equation for Γ should read:

$$\frac{d\Gamma}{dt} = -\frac{1}{\sqrt{1-e^2}}\frac{na}{\mu}\frac{\partial R}{\partial\theta} - \frac{\Gamma}{\sqrt{1-e^2}}\frac{na}{\mu}\left(\frac{\partial R}{\partial w} + \frac{\partial R}{\partial\varpi}\right),$$

i.e. the first term in the right member has the factor $\sqrt{1-e^2}$ in the denominator instead of in the numerator, and w replaces ϵ in the second term.

Other errors on p. 32 are :

The signs of the second terms of the right-hand members of (4) and both forms of (8) should be changed to minus. The divisor h' is to be changed to h'^2 in the last terms of (4), (5) and both forms of (8).

The last term of (9) should read:

$$-\frac{r'^2}{h'^2}h_\rho\frac{\partial F'}{\partial v'}.$$

The second term of the left member of (8) should read:

$$-\rho\left(\frac{dv}{dv'}\right)^2.$$

THE CORRELATION BETWEEN ABSOLUTE MAGNITUDE, LINEAR TANGENTIAL VELOCITY, DISTANCE, APPARENT MAGNITUDE AND PROPER MOTION.

Richard A. Robb, D.Sc., F.R.S.E.

1. Kapteyn * was the first to introduce mean parallax as an empirical function of (*a*) apparent magnitude, (*b*) apparent magnitude and proper motion. The latter function was of the form

$$\log \bar{\pi}_{m,\,\mu} = a + b \log \mu + cm.$$

This formula has been modified by van Rhijn † and Strömberg ‡ to take account of the fact that the mean parallax should not be zero when $\mu = 0$. Kapteyn's original formula was established on a theoretical basis by Schwarzschild,§ who discussed the frequency functions of apparent magnitude, absolute magnitude, proper motion, etc. Seares || extended the latter's work, and the results of his investigations will be given later.

When mean absolute magnitudes are considered, we may first give the formula of Wilson,¶

$$\bar{M} = a + bm + c \log (\bar{\mu} + k),$$

as it is the combination of Strömberg's formula for mean parallax and the formula $M = m + 5 + 5 \log \pi$. k is assumed to be constant and is equal to 0″·004. Luyten and Lundmark ** have considered the simpler forms

$$M = a + bH,$$
$$M = a + bH + cH^2,$$

where $H = m + 5 + 5 \log \mu$. The later investigations of Prasad †† and Cecchini ‡‡ use similar formulæ. There is a difference, however, in determining the mean absolute magnitude for stars with a certain H. For stars of types B8–A5, which will be discussed later, Prasad found that

$$M = -0{\cdot}43 + 0{\cdot}407\, H.$$

Cecchini considered the various sub-classes with the following results :—

Spectral Class	$M = a + bH$	
	a	b
B8	−1·00	+0·37
B9	−0·85	+0·36
A0	−0·70	+0·36
A2	−0·35	+0·38
A3	−0·25	+0·39
A5	−0·35	+0·42

* *G.P.*, 8, 1901. † *G.P.*, 34, 1923. ‡ *Ap. J.*, 47, 7, 1918.
§ *A.N.*, 190, 361, 1912. || *Ap. J.*, 59, 310, 1924. ¶ *Ap. J.*, 36, 49, 1925.
** *L.O.B.*, 336, 3, 1922 ; *Pub. A.S.P.*, 34, 156, 1922 ; 35, 209, 1923.
†† *M.N.*, 85, 157, 1924. ‡‡ *Pub. R.O. Merate* (*Como*), 5, 1931.

Gyllenberg * considers the correlation between M and m for A stars whose spectroscopic absolute magnitudes are known, and finds that, if $M = c + dm$, then $d = +0{\cdot}15$. Regarding the correlation between total linear tangential velocity (uncorrected for solar motion) and absolute magnitude reference may be made to a paper by Prasad.†

2. A survey of the relations between the various attributes of a star may be made by using the general formulæ of linear regression. These formulæ have the advantage of being independent of the frequency distributions of the attributes and may be expressed as follows:—

If a star has two attributes x and y measured from their respective arithmetic means, and there are n stars in the population considered, then the product of x and y for each of the n stars gives

$$\frac{\Sigma xy}{n} = r_{12}\sigma_1\sigma_2,$$

where linear regression is assumed between x and y, and r_{12}, σ_1, σ_2 are respectively the coefficient of correlation and standard deviations of x and y. The regression of y on x is then given by the formula

$$y = \frac{r_{12}\sigma_2}{\sigma_1}x,$$

where for a given x, y is the best value according to the method of least squares. Similarly the regression of x on y is given by

$$x = \frac{r_{12}\sigma_1}{\sigma_2}y.$$

Considering three variables x_1, x_2, x_3 measured from their means, the regression of x_3 on x_1 and x_2 is represented by the formula

$$x_3 = \frac{\sigma_3}{\sigma_1}\frac{r_{13} - r_{12}r_{23}}{1 - r_{12}^2}x_1 + \frac{\sigma_3}{\sigma_2}\frac{r_{23} - r_{12}r_{13}}{1 - r_{12}^2}x_2,$$

where σ_1, σ_2, σ_3 refer to the standard deviations of x_1, x_2, x_3 respectively, and r_{12}, r_{13}, r_{23} to the coefficients of correlation between x_1 and x_2, x_1 and x_3, x_2 and x_3 respectively. It may also be noted that two variables x and y, say, are independent when $r_{xy} = 0$.

The attributes of a star which will now be considered are (1) absolute magnitude M, (2) total linear tangential velocity P, (3) distance r, (4) apparent magnitude m, (5) proper motion μ, and (6) the characteristic

$$H = m + 5 + 5\log_{10}\mu.$$

The following relations hold :—

$$M = m + 5 - 5\log_{10} r, \qquad P = k\mu r, \qquad k = 4{\cdot}738.$$

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* *Lund Medd.*, **1**, 142, 1936.
† *M.N.*, 85, 417, 1925.

Here r is expressed in parsecs, μ in seconds of arc and P in km./sec. If

$$m_1 = m + 5, \qquad y = -5 \log r,$$
$$t = -5 \log \mu, \qquad T = -5 \log P + 3{\cdot}3780,$$

then denoting, for simplicity, the deviations of all the quantities from their means by the symbols M, m, y, T, t, H the following formulæ are obtained:—

$$M = m + y, \tag{1}$$

$$T = t + y, \tag{2}$$

$$H = m - t, \tag{3}$$

and hence

$$H = M - T. \tag{4}$$

3. It will now be assumed that linear regression exists between any pair of the six variables M, m, T, t, H, y, but that no correlation exists between M and y, T and y. These latter conditions, expressing the independence of absolute magnitude and distance, and of linear tangential velocity and distance, are usually assumed for stars of a given spectral type. It will be noted, from equation (4), that H is also now independent of distance. The following regression lines are easily derived :—

Regression between M and m,

$$M_m = \frac{\sigma_M^2}{\sigma_m^2} m. \tag{5}$$

Regression between T and t,

$$T_t = \frac{\sigma_T^2}{\sigma_t^2} t. \tag{6}$$

Regression between M and T,

$$M_T = \frac{1}{\sigma_T^2}(\sigma_m \sigma_t r_{mt} - \sigma_y^2) T. \tag{7}$$

Regression between M and H,

$$M_H = \frac{\sigma_m}{\sigma_H} r_{mH} \,.\, H. \tag{8}$$

Regression between T and H,

$$T_H = \frac{\sigma_t}{\sigma_H} r_{tH} \,.\, H. \tag{9}$$

Regression between y and m,

$$y_m = -\frac{\sigma_y^2}{\sigma_m^2} m. \tag{10}$$

Regression between y and t,

$$y_t = -\frac{\sigma_y^2}{\sigma_t^2} t. \tag{11}$$

Regression between M and t,

$$M_t = \left(\frac{\sigma_m}{\sigma_t} r_{mt} - \frac{\sigma_y^2}{\sigma_t^2} \right) t. \tag{12}$$

Regression between T and m,

$$T_m = \left(\frac{\sigma_t}{\sigma_m} r_{mt} - \frac{\sigma_y^2}{\sigma_m^2}\right) m. \tag{13}$$

Regression between M and T,

$$T_M = \frac{1}{\sigma_M^2}(\sigma_m \sigma_t r_{mt} - \sigma_y^2) M. \tag{14}$$

Regression between y and m, t,

$$y_{m,t} = -\frac{\sigma_y^2}{\sigma_m^2 \sigma_t (1 - r_{mt}^2)}(\sigma_t - r_{mt}\sigma_m) m,$$
$$-\frac{\sigma_y^2}{\sigma_m \sigma_t^2 (1 - r_{mt}^2)}(\sigma_m - r_{mt}\sigma_t) t. \tag{15}$$

Regression between M and H,

$$M_{H,m} = \frac{1}{\sigma_m^2 (1 - r_{mH}^2)}(\sigma_M^2 - \sigma_m^2 r_{mH}^2) m,$$
$$+\frac{r_{mH}}{\sigma_H \sigma_m (1 - r_{mH}^2)}(\sigma_m^2 - \sigma_M^2) H. \tag{16}$$

Regression between T and H, t,

$$T_{H,t} = \frac{1}{\sigma_t^2 (1 - r_{tH}^2)}(\sigma_T^2 - \sigma_t^2 r_{tH}^2) t,$$
$$+\frac{r_{tH}}{\sigma_H \sigma_t (1 - r_{tH}^2)}(\sigma_t^2 - \sigma_T^2) H. \tag{17}$$

Regression between T and H, m,

$$T_{H,m} = \frac{1}{\sigma_H} \frac{\sigma_t r_{Ht} - r_{Hm}\left(\sigma_t r_{mt} - \frac{\sigma_y^2}{\sigma_m}\right)}{1 - r_{Hm}^2} H,$$
$$+\frac{1}{\sigma_m} \frac{\sigma_t r_{mt} - \frac{\sigma_y^2}{\sigma_m} - r_{Hm}\sigma_t r_{Ht}}{1 - r_{Hm}^2} m. \tag{18}$$

Two points of interest may be noted. As has been stated, the characteristic H is independent of distance, and thus from equation (8) it appears feasible to obtain mean absolute magnitudes without the use of parallaxes. The quantities required depend only on the apparent magnitudes and proper motions, and it is necessary, of course, that the coefficient of H in equation (8) should differ significantly from zero. The regression of y on m, t (corresponding to Kapteyn's formula) depends again only on apparent magnitudes and proper motions apart from a constant factor σ_y^2.

4. As an illustration of the formulæ we shall use the data given by Seares,* viz. :—

$$\sigma_M = 2{\cdot}509, \qquad \sigma_y = \ \ 2{\cdot}879,$$
$$\sigma_T = 2{\cdot}632, \qquad r_{TM} = -0{\cdot}255.$$

* *Loc. cit.*

In deriving the correlation coefficient, a term $-0{\cdot}0019\ M^2$ was omitted from Seares's formula which gives the regression of T on M. We now obtain the following coefficients :—

$$\begin{aligned} \sigma_m &= \ 3{\cdot}819, & \sigma_t &= \ 3{\cdot}305, \\ r_{mt} &= +0{\cdot}574, & \sigma_H &= \ 3{\cdot}317, \\ r_{mM} &= +0{\cdot}657, & r_{MH} &= +0{\cdot}881, \\ r_{mH} &= +0{\cdot}579, & r_{TH} &= -0{\cdot}682, \\ r_{tH} &= -0{\cdot}335, \end{aligned}$$

and hence that

$$\begin{aligned} M_m &= +0{\cdot}43\ m, & M_H &= +0{\cdot}67\ H, \\ T_H &= -0{\cdot}33\ H, & y_{m,\,t} &= -0{\cdot}29\ m - 0{\cdot}57\ t, \\ T_{H,\,m} &= -0{\cdot}43\ H + 0{\cdot}15\ m. \end{aligned}$$

The formulæ obtained by Kapteyn, van Rhijn and Seares are as follows :—

$$T_H = -0{\cdot}36\ H \text{ (Kapteyn, van Rhijn and Seares).}$$
$$T_{H,\,m} = -0{\cdot}44\ H + 0{\cdot}15\ m \text{ (Seares).}$$

5. The assumption that M and T and hence also H are independent of y thus leads, in particular, to the formula

$$T_H = -0{\cdot}33\ H. \tag{19}$$

Seares,* however, in his investigation of stars of large proper motion for which $\mu \geqslant 0''{\cdot}500$, collected by Luyten, found that the coefficient of H had different values on either side of the limit $H = 12{\cdot}5$. Thus

$$T_H = -0{\cdot}33\ H, \qquad H < 12{\cdot}5, \tag{20}$$
$$T_H = -0{\cdot}03\ H, \qquad H > 12{\cdot}5. \tag{21}$$

From this fact the important observation is made that the stars of low luminosity for which $H > 12{\cdot}5$ have frequencies that increase to the limit of absolute magnitude discussed, that is to about $+14{\cdot}5$. It is not correct, however, to associate the coefficient of H in equation (19) with those in equations (20) and (21) for the following reason. The selection of stars according to proper motion will introduce a correlation between the characteristics H and y. If such a correlation is significant one cannot associate such stars with those for which H and y have been taken as independent. If we assume that there is linear regression between H and y we find that the regression of T on H is given by the equation

$$T_H = \left(\frac{\sigma_t r_{tH}}{\sigma_H} + \frac{\sigma_y r_{Hy}}{\sigma_H} \right) H, \tag{22}$$

obtained from equation (2) by multiplying by H and taking mean values. This differs from equation (9) in the term $\frac{\sigma_y r_{Hy}}{\sigma_H}$, which is introduced through the assumed correlation between H and y. We now consider the 50 stars of low luminosity tabulated by Seares, and we find that

* *Ap. J.*, 310, 1924.

$$\frac{\sigma_t r_{tH}}{\sigma_H} = -0{\cdot}70, \qquad \frac{\sigma_y r_{Hy}}{\sigma_H} = +0{\cdot}67, \qquad H > 12{\cdot}5.$$

The probable errors of these coefficients are $\pm 0{\cdot}030$ and $\pm 0{\cdot}113$ respectively. Again, from the faint stars of van Maanen and Willis and of Ross and Wolf, Seares * found that

$$T_H = -4{\cdot}865 - 0{\cdot}0875\, H, \qquad H > 12{\cdot}5.$$

From the revised parallaxes for these stars given by Seares, we find that the regression between H and y is given by

$$\frac{\sigma_y r_{Hy}}{\sigma_H} = +0{\cdot}52, \qquad H > 12{\cdot}5,$$

with a probable error $\pm 0{\cdot}021$. This coefficient is to be compared with the value $+0{\cdot}67$ found above. Finally for the stars of Luyten's catalogue for which $H < 12{\cdot}5$, we find that

$$\frac{\sigma_y r_{Hy}}{\sigma_H} = -0{\cdot}29 \pm 0{\cdot}006.$$

The probable errors show that there is a significant regression between H and y for the groups considered, and thus it is not possible to deduce the form of the luminosity function of stars of low luminosity from formulæ which assume the independence of H and y, as is done by Seares.

6. We now propose to examine the regression between M and H, dealing in particular with the A2 stars. Cecchini † found that for this sub-class

$$M_H = +0{\cdot}38\, H.$$

Thus if M and T are assumed independent of y, we should then have

$$\frac{\sigma_m r_{mH}}{\sigma_H} = \frac{\sigma_M r_{MH}}{\sigma_H} = +0{\cdot}38.$$

From Cecchini's ‡ general catalogue of parallaxes which was used by him to determine the regression of M on H, it was found that for 88 stars of type A2, contained also in Boss's *P.G.C.*,

$$\frac{\sigma_m r_{mH}}{\sigma_H} = +0{\cdot}05 \pm 0{\cdot}035,$$

this coefficient being calculated from apparent magnitudes and the characteristic H. As most of the stars were brighter than the 6th apparent magnitude an investigation of all stars of Boss's *P.G.C.* brighter than the 6th magnitude was made. For 305 A2 stars

$$\frac{\sigma_m r_{mH}}{\sigma_H} = +0{\cdot}03 \pm 0{\cdot}015.$$

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* *Ap. J.*, 74, 320, 1931.
† *Loc. cit.*
‡ *Pub. R.O. Merate* (*Como*), 4, 1930.

The discrepancy between the results for the regression of M on H may be explained, as before, by a correlation between H and y. Thus, if we use the formula

$$\frac{\sigma_m r_{mH}}{\sigma_H} = \frac{\sigma_m r_{mH}}{\sigma_H} + \frac{\sigma_y r_{Hy}}{\sigma_H},$$

which allows for this correlation, we find that, since

$$\frac{\sigma_M r_{MH}}{\sigma_H} = +0{\cdot}38, \qquad \frac{\sigma_m r_{mH}}{\sigma_H} = +0{\cdot}05,$$

then

$$\frac{\sigma_y r_{Hy}}{\sigma_H} = +0{\cdot}33 \pm 0{\cdot}051.$$

Investigation of the remaining sub-classes of type A reveal similar positive coefficients.

7. We have considered, in particular, the correlation of y with H, but, since $H = M - T$, such a correlation will reveal itself in a correlation between y and M or between y and T or between y with M and T. It is possible at once, for stars limited to a certain apparent magnitude, to construct a regression between y and M. Thus in the formula

$$M = m + 5 + 5 \log \pi,$$

let M vary from -3 to $+4$, and m between 1 and 5. A double-entry table may now be drawn up between M and $y = 5 \log \pi$. The table is represented in the figure, the possible values of y being shown by the shaded portion. The concentration of frequencies will be towards the left-hand edge of the shaded portion (where $m = 5$), and it is easily seen how regression arises out of the limitation in apparent magnitude. As the regression of M on y will depend to a large extent on the limits of apparent magnitude used, it is not desirable that any regression such as that of M on H, which is derived from such a coefficient, should be used as a means of obtaining mean absolute magnitudes. Another consequence of this correlation between M and y for stars limited to a certain apparent magnitude is that it will be no longer possible, for example, to represent the number of stars having absolute magnitudes between M and $M + dM$ situated in a shell of radius r and thickness dr by the formula

$$4\pi r^2 \Delta(r) dr \phi(M) dM,$$

where $\phi(M)dM$ and $\Delta(r)dr$ represent the luminosity and density functions respectively. The formula will require to be of the form

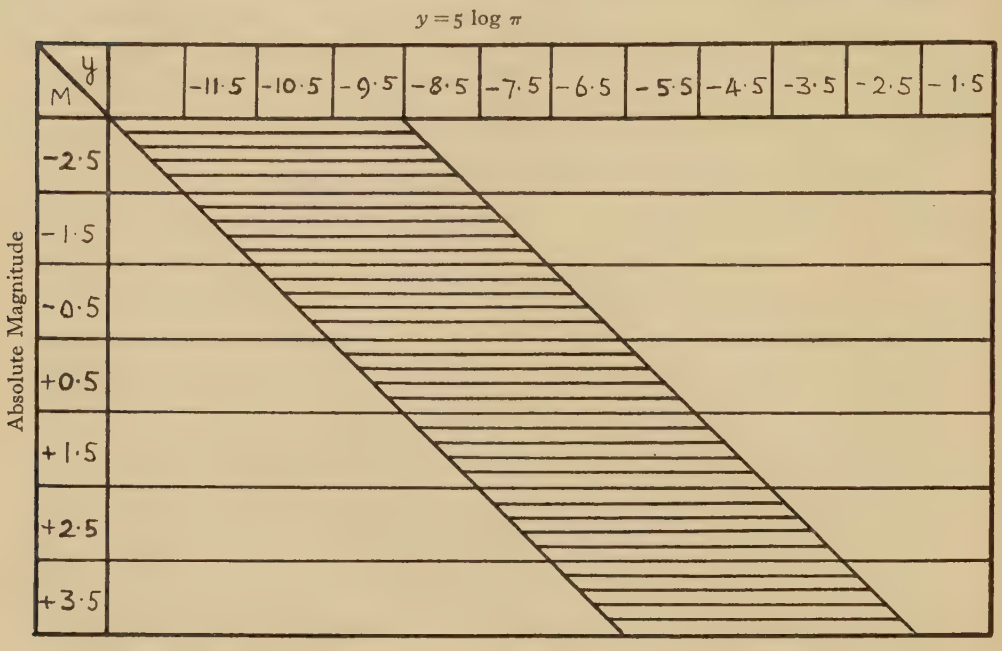
$$4\pi r^2 D(r, M) dr dM.$$

$D(r, M)drdM$ may, however, be conveniently represented by a normal correlation surface in $y = -5 \log r$ and M, as it is easily possible in such a surface to allow for linear regression between M and y.

8. The effect of accidental errors ϵ may also be investigated by means of regression lines. Let

$$M' = M - \epsilon,$$

where M' is the true magnitude of a quantity, M its observed magnitude, both being measured from the mean magnitude of the group under



consideration. We shall assume that ϵ and M' are independent of each other, and that the regression of ϵ on M is linear. We thus have

$$\sigma_\epsilon^2 = \sigma_M^2 - \sigma_{M'}^2,$$
$$\sigma_\epsilon = \sigma_M r_{M\epsilon},$$

and the regression of ϵ on M is given by

$$\epsilon_M = \frac{\sigma_\epsilon^2}{\sigma_M^2} M.$$

Now let us consider the regression between the logarithm of the linear tangential velocity and the absolute magnitude, where the logarithm of the parallax is subject to accidental error. With the same notation as above, let

$$M = m + y,$$

where all quantities are measured from their respective means, and M is the observed absolute magnitude corresponding to an observed parallax. Also let

$$M' = m + y',$$

where M' is the true absolute magnitude corresponding to the true parallax. Thus we have

$$M' - M = y' - y$$
$$= \epsilon.$$

Similarly

$$T = t + y, \qquad T' = t + y',$$

and

$$T' - T = \epsilon,$$

where, as before, T' and T refer to true and observed quantities respectively. It will be noticed that the errors in m and t have been neglected. We then obtain the equations

$$T' = t - m + M',$$
$$T = t - m + M,$$

and if we assume that

$$T' = a'M', \qquad T = aM,$$

we find that

$$r_{T'M'}\sigma_{T'}\sigma_{M'} - r_{TM}\sigma_T\sigma_M = \sigma_{M'}^2 - \sigma_M^2,$$

that is,

$$\frac{r_{T'M'}\sigma_{T'}}{\sigma_{M'}} = \frac{r_{TM}\sigma_T\sigma_M}{\sigma_{M'}^2} + 1 - \frac{\sigma_M^2}{\sigma_{M'}^2},$$

or

$$a' = \frac{\sigma_M^2}{\sigma_{M'}^2}a + 1 - \frac{\sigma_M^2}{\sigma_{M'}^2}.$$

This formula corresponds to that given by Seares.* The method may easily be extended to other variables provided that linear regression can be assumed.

NOTE ON MR. R. A. McINTOSH'S PAPER: "THE VELOCITIES OF METEOR STREAMS."

Rev. M. Davidson, B.A., D.Sc.

In *M.N.*, **96**, 704, 1936, Mr. R. A. McIntosh suggests a method for determining the velocities of meteors by measuring the displacement of the radiant in northern and southern latitudes. There is a fundamental error in the method which renders it useless, and this arises from the manner in which Table I is compiled.

Using Mr. McIntosh's symbols and noticing that $\tan \frac{1}{2}\Delta Z = \frac{1}{2}\Delta Z$ for all practical purposes, since $\frac{1}{2}\Delta Z$ is less than 10°,

$$\tfrac{1}{2}\Delta Z_N = \tan \tfrac{1}{2}Z_N \tan \tfrac{1}{2}\pi,$$
$$\tfrac{1}{2}\Delta Z_S = \tan \tfrac{1}{2}Z_S \tan \tfrac{1}{2}\pi,$$

from which

$$\tfrac{1}{2}\Delta Z = \tan \tfrac{1}{2}\pi \,(\tan \tfrac{1}{2}Z_N + \tan \tfrac{1}{2}Z_S),$$

where ΔZ is the total zenithal displacement—that is, the sum of the combined displacements. As it is the same shower, π is the same for both positions.

* *M.N.*, **84**, 15, 1923.

Mr. McIntosh has compiled Table I by using the formula

$$\tfrac{1}{2}\Delta Z = \tan \tfrac{1}{2}\pi \tan \tfrac{1}{2}(Z_N + Z_S),$$

which is obviously an error. If he had extended the table to values of combined zenith distances approaching 180°, he would have seen the utter futility of the table, because in this case he would have found a displacement approaching infinity. For small values of Z the method would be approximately correct, but for large values Table I is useless.

MEAN AREAS AND HELIOGRAPHIC LATITUDES OF SUNSPOTS IN THE YEAR 1935.

Royal Observatory, Greenwich.

(Communicated by the Astronomer Royal)

The following results are in continuation of those given in *M.N.*, **95**, 761, 1935, and are derived from the measurement at Greenwich of photographs taken at the Royal Observatories of Greenwich and the Cape and at the Kodaikanal Observatory, India.

Table I gives the mean daily areas of umbræ, whole spots and faculæ for each synodic rotation of the Sun included in the year 1935; the means for the year as a whole are included in Table II, which summarizes the yearly values from 1933–35.

TABLE I

No. of Rotation	Rotation Commenced U.T.	Days Photographed	Mean Daily Areas: Projected *			Mean Daily Areas: Corrected for Foreshortening †		
			Umbræ	Whole Spots	Faculæ	Umbræ	Whole Spots	Faculæ
1088	1935 Jan. 13·19	28	83	437	734	67	355	873
1089	Feb. 9·53	27	43	203	638	34	168	743
1090	Mar. 8·86	27	49	231	590	37	176	706
1091	Apr. 5·16	28	38	187	555	30	150	680
1092	May 2·42	27	94	503	582	70	374	690
1093	29·64	27	139	700	805	112	554	1007
1094	June 25·84	27	198	1077	915	153	809	1123
1095	July 23·04	27	61	291	895	50	238	1052
1096	Aug. 19·27	28	137	698	1136	116	608	1426
1097	Sept. 15·52	27	132	648	1284	105	526	1569
1098	Oct. 12·80	27	249	1391	1194	187	1047	1420
1099	Nov. 9·10	28	479	2629	1233	339	1862	1487
1100	Dec. 6·41	27	335	1826	1427	261	1452	1691

* Expressed in millionths of the Sun's disc.

† Expressed in millionths of the Sun's hemisphere.

TABLE II

Year	No. of Days		Mean Daily Areas					
			Projected *			Corrected for Foreshortening †		
	Photographed	Without Spots	Umbræ	Whole Spots	Faculæ	Umbræ	Whole Spots	Faculæ
1933	365	234	24	120	225	18	88	267
1934	365	145	27	142	297	22	119	354
1935	365	21	152	810	910	117	624	1100

* Expressed in millionths of the Sun's disc.
† Expressed in millionths of the Sun's hemisphere.

Table III gives for each rotation in the year 1935 the mean daily area of the whole spots (corrected for foreshortening) and the mean heliographic

TABLE III

No. of Rotation	Rotation Commenced U.T.	Spots North of the Equator		Spots South of the Equator		Mean Latitude of entire Spotted Area	Mean Distance from Equator of all Spots
		Mean Daily Area	Mean Heliographic Latitude	Mean Daily Area	Mean Heliographic Latitude		
			°		°	°	°
1088	1935 Jan. 13·19	85	27·60	271	21·75	−10·00	23·15
1089	Feb. 9·53	35	14·05	134	18·16	−11·53	17·31
1090	Mar. 8·86	117	13·62	59	19·75	+ 2·40	15·68
1091	Apr. 5·16	37	22·15	112	34·05	−20·01	31·08
1092	May 2·42	35	21·36	339	26·59	−22·14	26·11
1093	29·64	121	26·25	433	25·79	−14·42	25·89
1094	June 25·84	84	26·25	725	24·53	−19·27	24·70
1095	July 23·04	118	25·56	120	18·25	+ 3·48	21·87
1096	Aug. 19·27	506	25·20	102	17·85	+17·97	23·96
1097	Sept. 15·52	287	21·98	238	20·12	+ 2·88	21·14
1098	Oct. 12·80	701	21·00	345	26·61	+ 5·30	22·85
1099	Nov. 9·10	475	22·95	1386	23·43	−11·60	23·31
1100	Dec. 6·41	178	17·98	1275	22·30	−17·37	21·77

TABLE IV

Year	Days Photographed	Spots North of the Equator		Spots South of the Equator		Mean Latitude of entire Spotted Area	Mean Distance from Equator of all Spots
		Mean Daily Area	Mean Heliographic Latitude	Mean Daily Area	Mean Heliographic Latitude		
			°		°	°	°
1933	365	86	10·60	2	8·35	+10·23	10·56
1934	365	44	16·45	74	28·09	−11·49	23·75
1934	Old Cycle	19	4·10	1	4·37	+ 3·79	4·11
	New Cycle	26	25·44	74	28·31	−14·46	27·57
1935	365	205	22·42	419	23·72	− 8·57	23·30

latitude of the spotted areas for both northern and southern hemispheres. The mean heliographic latitude of the entire spotted area and the mean distance from the equator of all spots are also tabulated. The mean value for the year 1935 is included in Table IV.

Tables II and IV are in continuation of similar tables given in *Monthly Notices*: for the years 1874 to 1888—**49**, 381–382; 1889 to 1902—**63**, 465–466; 1901 to 1914—**76**, 402–403; 1913 to 1924—**85**, 1007–1008; 1923 to 1933—**94**, 870–871.

The rotations in Tables I and III are numbered in continuation of Carrington's series.

The chief features of the record of sunspots and faculæ for the year 1935 are as follows :—

1. The marked increase in the mean daily areas of sunspots and faculæ as shown in Tables I and II denotes that the new 11-year cycle is well established. The increase in activity was especially shown during the latter half of 1935, when half a dozen large groups of sunspots, each exceeding 500 millionths of the Sun's hemisphere in mean area, were recorded. One of these groups, with central meridian passage on December 2, was the largest group observed since 1928 September.

2. On 21 days no spots were recorded, as compared with 145 days in 1934 and 234 days in 1933 when the last minimum of the 11-year cycle occurred.

3. The southern hemisphere of the Sun was more active than the northern. The respective mean areas for the year were in the ratio of 2 : 1 (Table IV).

4. The mean latitude of all spots irrespective of sign is 23°·3. The occurrence of equatorial spots has nearly ceased, the number of spots nearer the equator than ± 10° being only 3 per cent. that of the total number of groups recorded during 1935. The highest latitude recorded for a group was 38°·1 and the lowest 0°·3.

5. The number of groups lasting for two days or longer was 173; of these 76 were in the northern hemisphere and 97 in the southern. In addition, there were 64 very small groups recorded on one day only; 29 were north and 35 south of the equator.

Royal Observatory, Greenwich :
1936 *October* 13.

STELLAR PARALLAXES DETERMINED PHOTOGRAPHICALLY AT THE CAPE OBSERVATORY. (EIGHTH LIST.)

(Communicated by H.M. Astronomer)

The form of publication is similar to that of previous lists in the *Monthly Notices*.

The stars have generally been selected on account of large proper-motion. The lists in future will contain a larger number of stars in the Cape Astrographic Zone, $-40°$ to $-52°$, including some rather faint stars. Also some stars have been included in the working programme as double stars with dynamical parallax exceeding $0''\cdot03$, or spectroscopic parallax of similar amount. The star No. 703 was included merely on account of spectral type K5.

Attention may be drawn to errors in published proper-motions.

No. 684.—The sign of the proper-motion in R.A. is given erroneously in U.O.C. 48.

No. 695.—Observed on the basis of an erroneous unpublished Cape proper-motion.

No. 728.—The proper-motion given in U.O.C. 19 appears to be seriously in error.

No. 741.—This star was observed in the belief that it was Wolf 629, and is the nearest star of the proper magnitude to the catalogued position of Wolf 629. It precedes Wolf 630 by 26^s. Wolf 629 is catalogued as preceding Wolf 630 by 34^s, but it should be $3^s\cdot4$. The error originated in *A.N.*, 4908, and was carried into *Heidelberg Publications*, 7, No. 10; *Lick Observatory Bull.* 344; and probably other lists of stars with large proper-motion. The error also appears in Schlesinger's *Parallax Catalogue* (1935 edition), although the parallaxes obtained at two observatories indicate that the proper-motion star was correctly identified. The correct star is not shown on the Cape series of plates as it falls under the occulting shutter. The R.A. of Wolf 629 should be $16^h\ 50^m\ 5^s$.

Nos. 743 *and* 744.—These two stars are C.P.D. $-44°$ 8684, 8685. The proper-motions given are from measurement of Cape Astrographic plates. The stars were observed on the strength of proper-motions given in U.O.C. 37, where the displacements in 21 years were given as about $7''$. The result indicates the possible errors from comparison of star images on the corners of photographic plates.

With regard to the names of the stars given in the list, no special system has been followed except in the Durchmusterung number, which is B.D. down to $-22°$, Co.D. from $-23°$ to $-51°$, and C.P.D. $-52°$ southwards. A reference is sometimes given to a publication where the proper-motion is given. The letters F.S. refer to a catalogue of faint stars, not yet published.

It includes stars of brightness 9·1 to 9·5 of the C.P.D. in the Cape Astrographic Zone −40° to −52°.

The proper-motion in R.A. given in the second line for each star is derived from the photographs ; the proper-motion in R.A. and Dec. in the first line is from meridian observations or other sources. The two values of probable error given in the last column refer to the parallax and photographic proper-motion in R.A. respectively.

The parallaxes of four stars exceed 0″·100, and for two other stars this value is closely approached. The parallaxes for Nos. 685, 717, viz. 0″·262 and 0″·191, are first determinations.

The parallaxes are relative to the mean of about four comparison stars which are generally between magnitude 11 and 12.

No.	Name and D.M. No.	Mag. Vis. Ptg.	Spec. Type	R.A. Dec. (1900)	Proper-Motion R.A. Unit 0″·001	Proper-Motion Dec. Unit 0″·001	P.E. Unit Weight. Unit 0″·001	Parallax Unit 0″·001	P.E. Unit 0″·001
				h m s					
651	−57° 145	9·4	...	0 38 8	0	− 640	30	+ 13	± 9
		11·1		−57° 29′·7	+ 148				± 10
652	−10° 164	8·11	F5	0 42 53	+ 178	+ 211	26	+ 35	± 10
		8·53		−9° 53′·9	+ 183				± 11
653	Boss 181	5·24	F5	0 45 7	− 235	− 225	25	+ 58	± 8
	−11° 153	5·66		−11° 11′·0	− 197				± 9
654	−69° 44	7·4	G5	1 11 36	+ 370	+ 100	24	+ 45	± 8
		8·2		−69° 21′·0	+ 342				± 8
655	κ^2 Tucanæ	4·96	F8	1 12 23	+ 397	+ 116	22	+ 39	± 8
	−69° 45	5·46		−69° 24′·4	+ 409				± 8
656	C.A.Z. −43° 8, 110	...	...	1 16 57	+1241	− 450	26	+ 60	± 8
	F.S. 380	11·3		−42° 10′·1	+1276				± 8
657	−18° 287	7·40	Go	1 37 22	+ 543	0	21	+ 31	± 8
		7·96		−18° 23′·7	+ 537				± 8
658	τ Ceti	3·65	Ko	1 39 25	−1719	+ 856	28	+288	± 10
	−16° 295	4·65		−16° 27′·8	−1741				± 11
659	Wash. A.G. 471	9·1	F8	1 46 51	+ 172	− 350	26	+ 18	± 9
	−17° 332	9·6		−16° 48′·2	+ 105				± 9
660	Alb. 465	9·2	G5	1 51 5	+ 122	− 275	19	+ 12	± 6
	−27° 665	9·8		−26° 58′·9	+ 73				± 6
661	Ci. 20, 147	6·89	G5	2 2 29	− 285	− 370	30	+ 28	± 11
	−1° 293	7·67		−1° 5′·2	− 255				± 10
662	Boss 596	5·80	G5	2 32 49	− 14	− 279	25	+ 43	± 9
	−35° 903	7·3		−35° 0′·3	− 9				± 8

No.	Name and D.M. No.	Mag. Vis. Ptg.	Spec. Type	R.A. Dec. (1900)	Proper-Motion R.A. Unit 0″·001	Proper-Motion Dec. Unit 0″·001	P.E. Unit Weight. Unit 0″·001	Parallax Unit 0″·001	P.E. Unit 0″·001
				h m s					
663	Ci. 20, 183	9·6	G5	2 41 4	+ 316	− 250	29	+ 32	± 10
	−17° 529	10·4		−16° 52′·6	+ 274				± 8
664	U.O.C. 53	...	...	3 10 2	+ 358	+ 301	29	+ 26	± 11
		13·5		−65° 23′·2	+ 387				± 10
665	Ci. 20, 232	8·1	Ko	3 20 4	− 239	− 780	30	+ 76	± 10
	−5° 642	9·1		−5° 41′·7	− 245				± 10
666	Ci. 20, 250	6·74	F8	3 35 17	+ 712	− 217	28	+ 15	± 10
	−3° 592	7·24		−3° 32′·1	+ 678				± 10
667	δ Eridani	3·72	Ko	3 38 27	− 93	+ 743	22	+ 97	± 9
	−10° 728	4·72		−10° 6′·1	− 108				± 9
668	β Reticuli	3·80	Ko	3 42 57	+ 298	+ 73	24	+ 70	± 8
	−65° 263	4·80		−65° 7′·3	+ 247				± 9
669	U.O.C. 53	8·01	Go	3 46 48	+ 342	+ 36	20	+ 40	± 7
	−65° 272	8·9		−65° 8′·6	+ 315				± 7
670	ι Reticuli	4·81	K5	3 59 41	+ 50	+ 97	27	− 17	± 10
	−61° 293	6·7		−61° 21′·5	+ 55				± 10
671	−46° 1364	9·9	...	4 14 21	+ 53	− 191	27	+ 23	± 9
	F.S. 1264	11·8		−46° 10′·3	+ 38				± 8
672	−47° 1383	6·18	F5	4 24 10	+ 44	− 267	30	+ 22	± 10
		6·6		−47° 9′·8	+ 64				± 11
673	R Z Eridani	(7·8)	A5	4 39 2	...	...	27	+ 10	± 10
	−10° 993	(9·0)		−10° 52′·2	+ 37				± 10
674	−44° 1676	9·9	...	4 39 53	− 77	− 176	34	+ 5	± 12
	F.S. 1425	11·2		−44° 22′·7	− 58				± 11
675	−42° 1677	9·5	...	4 51 30	+ 112	+ 226	25	+ 30	± 8
	F.S. 1512	10·5		−42° 44′·6	+ 100				± 8
676	L 131–6	...	...	4 54 35	+ 939	− 610	25	+ 46	± 9
	A.J. 979	13·0		−61° 18′·2	+ 907				± 9
677	−43° 1726*f*	10·1	G	5 8 47	+ 48	− 191	26	+ 12	± 9
	F.S. 1649	10·9		−42° 59′·8	+ 71				± 10
678	−3° 1061	8·6	Ko	5 14 7	+ 704	+ 150	27	+ 22	± 9
	Ci. 20, 327	9·6		−3° 10′·9	+ 702				± 8
679	β Leporis	2·96	Go	5 23 58	+ 6	− 94	26	+ 33	± 10
	−20° 1096	3·52		−20° 50′·3	− 2				± 10

No.	Name and D.M. No.	Mag. Vis. Ptg.	Spec. Type	R.A. Dec. (1900)	Proper-Motion R.A. Unit 0″·001	Proper-Motion Dec. Unit 0″·001	P.E. Unit Weight. Unit 0″·001	Parallax Unit 0″·001	P.E. Unit 0″·001
				h m s					
680	α Leporis	2·69	Fo	5 28 19	+ 1	+ 3	32	+ 30	± 12
	−17° 1166	2·97		−17° 53′·6	+ 30				± 12
681	β Pictoris	3·94	A3	5 44 55	+ 7	+ 90	26	+ 59	± 10
	−51° 1620	4·02		−51° 6′·2	+ 13				± 10
682	−70° 447	8·34	Go	5 45 41	− 351	+1210	31	+ 69	± 10
	Ci. 18, 734	9·4		−70° 12′·8	− 307				± 9
683	δ Leporis	3·90	Ko	5 47 1	+ 241	− 653	27	+ 44	± 10
	−20° 1211	4·90		−20° 53′·3	+ 233				± 9
684	U.O.C. 48	...	...	6 2 20	− 626	− 315	31	+ 28	± 12
		13·5		−55° 18′·5	+ 695				± 10
685	Ci. 20, 390	...	...	6 24 18	+ 734	− 630	24	+262	± 10
	Ross 614	12·6		−2° 43′·8	+ 658				± 10
686	−44° 2691	9·9	...	6 26 33	+ 164	− 266	22	+ 16	± 7
	F.S. 2256	10·3		−44° 10′·5	+ 135				± 8
687	A.J. 852–3, 56	...	...	7 12 53	− 152	− 359	29	+ 42	± 11
		13·5		−13° 49′·2	− 136				± 12
688	−52° 1153*p*	6·36	F2	7 17 57	...	...	20	+ 38	± 7
		6·5		−52° 7′·7	− 58				± 6
	−52° 1153*f*	6·99			...	...	25	+ 31	± 8
					− 38				± 8
	Mean				...	...	...	+ 35	± 5
					− 48				± 5
689	−2° 2197	7·10	G5	7 31 3	− 294	+ 115	25	+ 37	± 9
	Ci. 18, 911	7·88		−2° 56′·0	− 266				± 10
690	−53° 1433	8·0	G5	7 45 27	+ 164	− 315	31	+ 22	± 10
		8·6		−54° 0′·4	+ 148				± 8
691	−59° 907	7·2	G5	7 47 34	+ 273	− 197	32	+ 40	± 11
	U.O.C. 48	8·1		−59° 7′·2	+ 307				± 9
692	−13° 2270	6·87	Ko	7 47 50	+ 28	− 85	29	+ 4	± 10
		7·87		−13° 36′·2	+ 41				± 8
693	−14 2293	7·9	Go	7 52 37	− 234	+ 207	24	+ 27	± 9
		8·5		−14° 19′·7	− 259				± 9
694	Boss 2224	5·86	Fo	8 18 36	− 11	+ 13	26	− 12	± 9
	−25° 5988	6·0		−26° 1′·7	+ 31				± 9

No.	Name and D.M. No.	Mag. Vis. Ptg.	Spec. Type	R.A. Dec. (1900)	Proper-Motion R.A. Unit 0″·001	Proper-Motion Dec. Unit 0″·001	P.E. Unit Weight. Unit 0″·001	Parallax Unit 0″·001	P.E. Unit 0″·001
				h m s					
695	C.A.Z. −43° 52,219	...	...	8 31 9	...	...	26	− 23	± 9
		11·3		−43° 13′·2	+ 3				± 9
696	Boss 2321	4·98	Ko	8 37 5	− 4	− 99	26	+ 35	± 8
	−15° 2554	5·98		−15° 35′·0	− 34				± 8
697	−66° 938	8·5	G5	8 52 37	− 308	+ 94	26	+ 36	± 8
	U.O.C. 53	9·3		−66° 25′·3	− 299				± 8
698	−41° 4886	...	...	9 8 5	− 249	+ 178	31	+ 25	± 11
	F.S. 5688	10·1		−41° 53′·4	− 228				± 10
699	−57° 1976	8·7	K2	9 15 20	− 138	+ 470	24	+ 28	± 9
	Ci. 20, 518	9·8		−57° 21′·6	− 100				± 8
700	−31° 7195	9·7	Ko	9 18 12	− 612	+ 280	22	+ 28	± 8
	Ci. 20, 521	9·3		−31° 43′·7	− 617				± 6
701	−12° 2889	9·5	G5	9 20 13	+ 762	− 530	16	+ 28	± 6
	Ci. 20, 524	10·3		−12° 31′·2	+ 557				± 6
702	−80° 349	9·8	...	9 27 17	+ 201	+1100	27	+ 54	± 11
	Ci. 20, 530	10·5		−80° 6′·9	+ 106				± 9
703	−72° 860	8·1	K5	9 42 52	...	...	23	− 12	± 8
		9·3		−72° 50′·5	− 14				± 7
704	−53° 3099	...	...	9 54 15	− 316	+ 208	33	+ 20	± 12
	U.O.C. 48	9·5		−53° 55′·8	− 333				± 9
705	−45° 5627	10	...	9 54 34	+ 497	− 463	32	+ 16	± 12
	C.A.Z. −46° 60,674	11·4		−45° 56′·0	+ 523				± 10
706	−16° 3030	9·1	G5	10 16 30	− 468	+ 120	28	+ 11	± 10
	Ci. 20, 576	9·9		−16° 32′·9	− 449				± 8
707	μ Hydræ	4·06	K5	10 21 15	− 128	− 84	28	+ 28	± 10
	−16° 3052	5·24		−16° 19′·5	− 144				± 8
708	Boss 2830	4·06	F2	10 33 6	− 164	− 30	22	+ 32	± 6
	−47° 6042	4·40	A3	−47° 42′·4	− 145				± 6
709	Boss 2846	6·40	Ko	10 36 19	− 141	− 129	28	+ 22	± 10
	−0° 2364	7·40		−1° 13′·0	− 103				± 10
710	−48° 5906	9·2	G5	10 42 2	− 381	+ 126	27	0	± 9
		9·8		−48° 51′·9	− 331				± 8
711	α Crateris	4·20	Ko	10 54 54	− 466	+ 121	24	+ 13	± 7
	−17° 3273	5·20		−17° 46′·0	− 456				± 7

No.	Name and D.M. No.	Mag. Vis. Ptg.	Spec. Type	R.A. Dec. (1900)	Proper-Motion R.A. Unit 0″·001	Proper-Motion Dec. Unit 0″·001	P.E. Unit Weight. Unit 0″·001	Parallax Unit 0″·001	P.E. Unit 0″·001
				h m s					
712	−64° 1636	8·3	F5	11 6 55	+ 220	− 272	25	− 16	± 8
	U.O.C. 53	8·7		−64° 52′·7	+ 195				± 9
713	−33° 7566	8·3	Go	11 8 4	+ 287	− 69	26	+ 20	± 7
	Alb. 3749	9·1		−33° 51′·7	+ 314				± 6
714	−1° 2505	9·2	...	11 12 11	− 525	− 10	23	+ 58	± 8
	Ci. 20, 626	10·8		−1° 26′·1	− 559				± 8
715	γ Crateris	4·14	A5	11 19 53	− 108	− 3	23	+ 12	± 8
	−16° 3244	4·28		−17° 8′·1	− 83				± 8
716	−31° 9113	9·1	...	11 30 30	− 114	− 880	28	+ 70	± 9
	Ci. 20, 649	11·0		−31° 57′·8	− 40				± 10
717	U.O.C. 37	...	...	11 40 30	+2653	− 440	22	+191	± 8
		11·3		−64° 16′·9	+2707				± 8
718	Boss 3095	5·04	G5	11 41 45	−1526	+ 387	30	+ 91	± 8
	−39° 7301	5·82		−39° 57′·4	−1532				± 8
719	−65° 1714	...	...	11 42 12	− 535	− 130	29	+ 10	± 9
		10·5		−65° 12′·1	− 486				± 10
	U.O.C. 37	...	...	11 42 14	− 535	− 130	30	− 4	± 9
		11·0		−65° 12′·3	− 496				± 10
	Mean				− 535	− 130	...	+ 3	± 6
					− 491				± 7
720	−41° 6879	9·5	G5	11 53 30	− 784	− 278	23	+ 6	± 7
		9·7		−41° 21′·5	− 769				± 8
721	−28° 9277	7·8	Go	12 0 4	+ 177	− 440	27	+ 53	± 10
		8·4		−28° 8′·9	+ 241				± 11
722	−66° 1752	6·38	Ko	12 17 51	− 780	+ 241	27	+ 7	± 10
	Ci. 18, 1556	7·6		−67° 5′·0	− 756				± 11
723	−48° 7414	10	...	12 20 42	− 595	− 119	27	+ 28	± 9
	F.S. 9945	12·1		−48° 17′·8	− 584				± 9
724	−48° 7426	6·22	Go	12 21 31	− 635	− 80	22	+ 25	± 7
		7·5		−48° 21′·4	− 631				± 7
725	δ Corvi	3·11	Ao	12 24 41	− 208	− 143	20	+ 21	± 7
	−15° 3482	3·11		−15° 57′·5	− 212				± 8
726	−77° 859	9·0	K5	12 33 33	− 759	+ 340	25	+ 32	± 9
		10·2		−77° 18′·2	− 866				± 8

No.	Name and D.M. No.	Mag. Vis. Ptg.	Spec. Type	R.A. Dec. (1900)	Proper-Motion R.A. Unit 0″·001	Proper-Motion Dec. Unit 0″·001	P.E. Unit Weight. Unit 0″·001	Parallax Unit 0″·001	P.E. Unit 0″·001
				h m s					
727	−55° 5262	8·3	Go	12 46 33	− 577	− 161	26	+ 14	± 8
	U.O.C. 48	9·1		−56° 1′·2	− 640				± 9
728	−53° 5416	8·1	G5	12 53 59	− 310	0	26	− 5	± 8
	U.O.C. 19	8·9		−53° 50′·3	+ 62				± 9
729	Boss 3427	5·04	F8	13 8 4	− 256	− 175	23	+ 50	± 6
	−58° 4740	6·0		−58° 34′·1	− 249				± 7
730	Boss 3448	4·80	G5	13 13 10	−1077	−1084	23	+ 88	± 8
	−17° 3813	5·58		−17° 45′·3	−1056				± 9
731	−26° 9804	9·8	Ko	13 28 55	− 560	− 161	31	− 11	± 9
		9·9		−26° 59′·4	− 555				± 10
732	−21° 3781	7·9	K5	13 44 27	−1692	− 505	26	+ 58	± 8
		9·4		−21° 35′·9	−1689				± 10
733	−23°11329	6·48	Go	13 45 50	− 589	− 293	18	+ 45	± 6
		7·1		−23° 53′·2	− 550				± 7
734	−44° 9595	9·8	...	14 40 9	− 73	− 440	22	+ 1	± 7
	F.S. 12408	10·9		−44° 58′·2	− 82				± 7
735	Boss 3797	6·43	Go	14 46 36	− 341	− 26	28	+ 16	± 8
	−30° 11780	7·3		−30° 9′·9	− 348				± 9
736	−10° 4011	10·0	K5	14 55 18	+ 28	− 466	26	+ 37	± 8
		11·2		−10° 43′·5	− 3				± 9
737	Wolf 562	9·2	...	15 14 14	−1309	− 231	32	+146	± 9
	−7° 4003	12·0		−7° 21′·4	−1214				± 11
738	U.O.C. 48	...	...	15 39 57	− 625	− 298	30	+ 16	± 8
		13		−57° 52′·0	− 502				± 9
739	s.f. star	...	...	15 40 4	...	...	26	− 6	± 7
		13		−57° 52′·2	+ 10				± 8
740	−44° 10577	10·6	Go	15 55 21	− 90	− 178	24	+ 8	± 7
		10·4		−44° 23′·8	− 132				± 8
741	Precedes Wolf 630	...	...	16 49 42	...	...	30	− 1	± 10
		13		−8° 8′·3	+ 8				± 12
742	−75° 1368	6·98	Go	17 9 11	− 974	− 80	18	+ 32	± 5
		8·1		−75° 14′·0	− 964				± 5
743	−44° 11967*p*	...	...	17 33 28	− 10	− 14	18	− 5	± 5
	F.S. 17096	10·7		−44° 53′·9	− 10				± 6

No.	Name and D.M. No.	Mag. Vis. Ptg.	Spec. Type	R.A. Dec. (1900)	Proper-Motion R.A. Unit 0″·001	Proper-Motion Dec. Unit 0″·001	P.E. Unit Weight. Unit 0″·001	Parallax Unit 0″·001	P.E. Unit 0″·001
				h m s					
744	−44° 11967*f*	8·32	B5	17 33 29	− 9	− 9	23	− 2	± 6
		8·7		−44° 54′·4	+ 12				± 7
745	−48° 12357	10	...	18 6 25	− 237	− 165	23	− 10	± 7
	F.S. 17862	11·6		−48° 57′·5	− 191				± 8
746	−51° 12475	7·4	F2	20 11 14	+ 67	+ 220	27	+ 4	± 8
		8·5		−51° 49′·3	+ 34				± 8
747	Co.D. −58° 7734	10	...	20 14 28	+ 818	− 225	27	+ 40	± 9
		11·5		−58° 35′·7	+ 755				± 9
748	−24° 16193	6·33	G5	20 34 15	+ 476	+ 454	34	+ 72	± 9
		7·0		−24° 8′·4	+ 500				± 12
749	Co.D. −54° 8749	...	...	20 34 30	0	− 356	17	+ 1	± 5
		12·5		−54° 21′·0	+ 74				± 7
750	*γ* Pavonis	4·30	F8	21 18 11	+ 82	+ 811	26	+ 98	± 8
	−65° 3918	4·80		−65° 49′·1	+ 105				± 10

STELLAR PARALLAXES DETERMINED AT THE ROYAL OBSERVATORY, GREENWICH.

(Communicated by the Astronomer Royal)

The following list is a continuation of the results obtained with the 26-inch Thompson refractor, and published annually in the November number of the *Monthly Notices*.

The visual magnitude and spectral type for stars contained in the *Draper Catalogue* have been extracted from that catalogue. The photographic magnitudes are from the *Greenwich Astrographic Catalogue*.

Two values of the proper motion in R.A. are given. The first (as also the proper motion in Dec.) is absolute, and is based on meridian observations or the comparison of long-interval plates. The second is the relative proper motion derived from the parallax plates.

In the last column the first value of the probable error refers to the parallax, and the second to the proper motion in R.A. These are based on the value of the probable error for a plate of unit weight given in the eighth column. The mean probable error for unit weight is ± 0″·026.

To minimise the effect of *quasi*-seasonal error the observations have been spread over not less than six epochs. The average number of plates per star is 28 ; for those stars observed during the summer months the number is in excess of the average to compensate for the loss of factor.

Generally six comparison stars of mean magnitude 11–12 have been used, the magnitude of the parallax star being reduced to equality by means of a rotating sector. For the brighter stars a neutral-tinted filter absorbing 4·8 magnitudes is used in conjunction with the sector. Two exposures of 3 minutes each are made on each plate, the carrier being turned through 180° after the first exposure. As the filter is permanently mounted on one of the carriers the reversal should eliminate any distortion due to the filter as well as to film displacement.

No.	Name and B.D. No.	Mag. Vis. Phot.	Type	R.A. Dec. (1900)	Proper Motion R.A.	Proper Motion Dec.	P.E. Unit Weight. Unit	Parallax. Unit	P.E. Unit
				h m s	Unit o″·001		o″·001	o″·001	o″·001
1	C. 958	8·6	G5	6 50 12	− 17	− 26	27	−12	± 8
	86° 96	8·9		+86° 29′	− 21				± 3
2	L.F. 1049	7·4	Go	7 17 31	− 28	−114	22	+24	± 5
	78° 254	7·6		+78° 54′	− 32				± 3
3	Gr. 1334	7·3	Ko	7 32 45	− 6	− 58	24	+ 2	± 7
	69° 436	7·8		+69° 34′	− 18				± 7
4	Gr. 1747	7·1	Ko	11 8 48	− 72	− 34	28	+ 4	± 8
	79° 356	7·9		+78° 51′	− 57				± 4
5	Gr. 1863	5·9	Ko	12 10 23	− 20	− 30	23	+ 9	± 7
	71° 610	6·8		+70° 45′	− 6				± 4
6	L.F. 2122	7·4	Ko	12 35 32	+ 5	− 41	27	+ 9	± 8
	73° 561	7·9		+73° 33′	+ 22				± 5
7	Gr. 2005	7·5	Ko	13 25 16	− 2	+ 1	26	+ 3	± 9
	70° 741	8·0		+70° 39′	+ 18				± 6
8	A.C. 70° 5824	...		13 36 27	−398	− 22	35	+25	± 11
		12·1		+70° 48′	−387				± 5
9	Gr. 2070	8·0	Ko	13 55 42	+ 14	+ 4	28	−18	± 9
	65° 966	7·7		+65° 23′	− 12				± 5
10	C. 2333	9·0	Ko	15 4 50	+ 16	+ 20	24	−16	± 8
	88° 90	9·3		+88° 24′	+ 48				± 2
11	14 UMi	7·4	F5	15 22 17	−108	+108	26	+19	± 8
	73° 672	7·5		+73° 50′	− 98				± 3
12	15 UMi	5·3	K5	15 34 23	− 46	+ 13	21	+ 1	± 7
	77° 592	6·5		+77° 41′	− 43				± 3
13	C. 2376	8·8	Ko	15 44 23	− 38	+ 32	29	− 7	± 9
	82° 464	9·1		+82° 9′	− 50				± 4

No.	Name and B.D. No.	Mag. Vis. Phot.	Type	R.A. Dec. (1900)	Proper Motion R.A. Unit 0″·001	Proper Motion Dec. Unit 0″·001	P.E. Unit Weight. Unit 0″·001	Parallax Unit 0″·001	P.E. Unit 0″·001
				h m s					
14	Gr. 2338	6·5	Ko	16 18 12	− 40	+ 47	33	− 8	± 12
	68° 868	7·5		+68° 48′	− 8				± 5
15	O.A. 16408	6·5	Ko	16 33 0	− 41	+ 70	29	+ 7	± 9
	72° 734	7·2		+72° 49′	− 32				± 4
16	L.F. 2818	8·5	Ko	16 37 52	+ 1	− 10	19	0	± 7
	72° 740	9·0		+72° 34′	+ 9				± 4
17	79° 515	8·5	Ko	16 50 3	− 4	+ 53	22	− 9	± 7
		9·0		+79° 41′	− 22				± 3
18	P. XVI 264	8·2	Ko	16 51 3	+ 25	− 64	25	+16	± 9
	65° 1150	8·5		+65° 22′	+ 19				± 3
19	Gr. 2412	6·5	Ko	16 59 36	− 22	− 44	25	+12	± 8
	69° 884	6·7		+69° 20′	− 1				± 4
20	72° 770	9·0	Ko	17 10 51	+ 20	+ 54	24	0	± 8
		9·1		+72° 0′	+ 26				± 4
21	Gr. 2476	7·5	F8	17 32 39	+ 73	−116	25	+19	± 9
	83° 512	7·8		+83° 25′	+ 71				± 5
22	Gr. 2655	5·8	Ko	18 34 35	+ 2	− 7	25	+ 9	± 10
	77° 699	6·6		+77° 28′	− 3				± 4
23	P. XVIII 186	7·3	Ko	18 36 4	+ 5	+ 1	25	− 5	± 11
	72° 852	8·1		+72° 20′	− 9				± 4
24	L.F. 3418	8·3	Ko	20 11 52	− 56	− 39	19	− 4	± 7
	72° 941	8·9		+72° 22′	− 55				± 5
25	Br. 2701	6·9	Ko	20 33 9	+ 43	+ 23	27	+27	± 10
	80° 657	7·3		+81° 6′	+ 51				± 5

THE FRANKLIN-ADAMS CHART.

Third Edition.

(Communicated by the Photographic Committee)

The Franklin-Adams Chart plates, 206 in number, afford a photographic representation of the whole sky on a uniform system.

The work owes its origin to a suggestion by Sir David Gill, pointing out the desirability of a photographic research into the structure of the Milky Way. Mr. Franklin-Adams subsequently extended the scheme to the photographic charting of the whole sky.

With this end in view he obtained from Messrs. Cooke & Sons in 1898 a 6-inch lens of the three-piece type working at f 4·5, designed by Mr. Dennis Taylor to give good definition over a large field. This lens was so successful that a larger one of similar type was ordered ; it was delivered in 1903 and was of 10 inches aperture and 45 inches focal length, giving very good images over a field $15° \times 15°$ on a scale of 20 mm. to the degree.

Mr. Franklin-Adams and his assistant, Mr. Kennedy, used this lens at the Cape Observatory in the years 1903–4, and the Southern sky was photographed with exposures of 2 hours for each plate. After his return to England he built an observatory adjoining his house at Mervel Hill, near Godalming. During the years 1905–9 the Northern sky was photographed, the exposure being increased to 2 hours 20 minutes for each plate. At the same time experiments were made towards the enumeration and classification of the stars, and to ascertain the best means of reproducing the Charts on paper. In the course of this work it was found that the Northern plates were so superior to those taken at the Cape that Mr. Franklin-Adams decided to repeat the Southern series. Illness prevented him from undertaking this work himself, and in 1909 he presented his 10-inch lens and equatorial to the Transvaal Observatory (now the Union Observatory), and arranged for his assistant, Mr. Mitchell, to go to Johannesburg in December of that year. The new series of plates was begun in April 1910. Mr. Mitchell was unable to complete the whole of the Southern plates, and after his return to England the series was completed by Mr. Wood under the direction of Mr. Innes.

Failing health rendered it impossible for Mr. Franklin-Adams to complete the two projects he had in mind : the publication of the photographs and a statistical discussion of the number of stars of different magnitudes in different parts of the sky. It was arranged that the statistical discussion should be made at Greenwich, and the plates were presented to the Royal Observatory. The results of this discussion are published in *Memoirs of the Royal Astronomical Society*, vol. lx. As regards the publication of the photographs, a limited edition of photographic copies of the plates, on a slightly reduced scale, was produced at the Royal Observatory and issued by the Royal Astronomical Society in 1914. A second edition was issued in 1922, and, in response to numerous applications, a third edition has now been produced.

It had been Mr. Franklin-Adams's intention to use, with his proposed atlas, transparent scales with the meridians and parallels of declination marked on them in their conic curves. A set of 7 such scales on celluloid, appropriate to the several declination centres, is issued with the present charts.

On the original plates, near the centre, the limiting magnitude of stars shown is about 17·5.

There is, however, a loss of about one-tenth of a magnitude for each degree progressing outwards from the centre. Further, in the reproduction about half a magnitude has been lost.

Particulars of the plates are given in the table below.

The Greenwich Mean Time (commencing at noon) of mid exposure follows the date wherever it was to be found in the records now in existence. When not given, it may be assumed that the plate was taken when the field was on or near the meridian on the evening of the date quoted.

The estimated number of stars in the field $15° \times 15°$ is taken from the table in *Memoirs of the Royal Astronomical Society*, vol. lx. When the number occurs in brackets the count was made on a plate other than that reproduced. Those with an asterisk were counted subsequently.

In the sixth column, J., C. and M.H. indicate Johannesburg, Cape of Good Hope and Mervel Hill respectively.

FRANKLIN-ADAMS CHART

Positions of Plate Centres, Dates of Taking and Estimated Number of Stars

No.	Centre			No. of Thousands of Stars in 225 sq. deg.	Place and Date of Taking (G.M.T. reckoned from Noon)		
	Dec.	R.A.	Gal. Lat.				
	°	h m	°				h m
1	−90	0 0	−27	415	J. 1910	May 8	8 27
2	−75	0 0	−42	351	J. 1910	July 14	14 51
3		2 40	−40	435	J. 1911	Jan. 29	
4		5 20	−32	(557)	C. 1904	Jan. 21	
5		8 0	−22	(606)	C. 1904	Feb. 8	
6		10 40	−14	717	J. 1910	May 9	
7		13 20	−13	695	J. 1910	May 3	9 28
8		16 0	−17	698	J. 1910	May 5	11 40
9		18 40	−26	600	J. 1910	July 12	9 54
10		21 20	−36	525	J. 1910	July 29	10 33
11	−60	0 0	−57	570	J. 1910	Aug. 10	13 5
12		1 36	−56	(481)	C. 1903	Nov. 17	
13		3 12	−48	320	J. 1911	Jan. 29	
14		4 48	−37	507	C. 1903	Dec. 13	
15		6 24	−25	644	J. 1911	Mar. 23	
16		8 0	−14	593	J. 1911	Apr. 20	
17		9 36	− 5	1281	J. 1912	May 10	5 45
18		11 12	+ 1	(1240)	J. 1910	May 7	6 17
19		12 48	+ 3	1024	J. 1910	May 2	8 43
20		14 24	0	1148	J. 1910	Apr. 10	12 8
21		16 0	− 6	1646	J. 1910	June 1	10 12

No.	Centre			No. of Thousands of Stars in 225 sq. deg.	Place and Date of Taking (G.M.T. reckoned from Noon)		
	Dec.	R.A.	Gal. Lat.				
	°	h m	°				h m
22	−60	17 36	−16	977	J. 1910	June 12	10 22
23		19 12	−27	692	J. 1910	July 30	8 50
24		20 48	−39	(761)	J. 1910	July 9	11 25
25		22 24	−50	429	J. 1910	Aug. 4	11 43
26	−45	0 0	−70	237	J. 1910	Sept. 1	11 22
27		1 20	−71	(202)	C. 1903	Nov. 18	
28		2 40	−60	214	J. 1911	Jan. 1	
29		4 0	−47	356	J. 1910	Oct. 3	13 15
30		5 20	−33	495	J. 1912	Mar. 8	6 59
31		6 40	−19	(629)	C. 1903	Dec. 26	
32		8 0	− 6	815	J. 1912	Apr. 20	6 5
33		9 20	+ 4	996	J. 1912	May 9	5 40
34		10 40	+13	692	J. 1910	May 5	
35		12 0	+17	701	J. 1910	May 10	7 43
36		13 20	+17	765	J. 1910	May 7	8 44
37		14 40	+13	881	J. 1910	May 10	10 5
38		16 0	+ 5	1063	J. 1910	May 6	11 22
39		17 20	− 6	1187	J. 1910	May 31	10 23
40		18 40	−19	745	J. 1910	May 16	13 30
41		20 0	−33	549	J. 1910	Aug. 4	9 19
42		21 20	−46	372	J. 1910	July 30	11 19
43		22 40	−60	375	J. 1910	Sept. 2	9 55
44	−30	0 0	−80	211	J. 1910	Sept. 26	10 20
45		1 0	−85	183	J. 1910	Nov. 1	8 33
46		2 0	−72	214	J. 1910	Nov. 24	
47		3 0	−60	382	J. 1911	Dec. 19	7 38
48		4 0	−47	372	J. 1911	Jan. 11	7 22
49		5 0	−34	460	J. 1911	Feb. 28	
50		6 0	−22	714	J. 1912	Mar. 13	6 48
51		7 0	−10	(1027)	C. 1903	Dec. 24	
52		8 0	+ 2	1326	J. 1912	Apr. 19	6 8
53		9 0	+12	759	J. 1912	May 7	6 24
54		10 0	+21	644	J. 1911	May 26	5 57
55		11 0	+28	424	J. 1911	May 30	5 58
56		12 0	+32	488	J. 1910	May 29	6 26
57		13 0	+33	554	J. 1910	May 6	8 35
58		14 0	+30	(539)	J. 1910	June 1	7 54
59		15 0	+24	682	J. 1910	May 30	8 43
60		16 0	+16	799	J. 1910	Aug. 1	6 8
61		17 0	+ 6	1115	J. 1910	Aug. 5	5 58
62		18 0	− 5	2017	J. 1910	July 29	8 2
63		19 0	−17	(838)	C. 1903	Sept. 17	
64		20 0	−29	567	J. 1910	Sept. 23	6 28
65		21 0	−42	537	J. 1910	Sept. 5	8 22
66		22 0	−55	299	J. 1910	Sept. 4	9 25
67		23 0	−68	300	J. 1910	Sept. 5	10 42
68	−15	0 0	−74	274	J. 1910	Sept. 10	10 53
69		1 0	−77	313	J. 1910	Oct. 3	10 50

No.	Centre			No. of Thousands of Stars in 225 sq. deg.	Place and Date of Taking (G.M.T. reckoned from Noon)		
	Dec.	R.A.	Gal. Lat.				
	°	h m	°				h m
70	−15	2 0	−68	332	J. 1910	Oct. 4	11 20
71		3 0	−56	233	J. 1911	Dec. 13	7 59
72		4 0	−42	402	J. 1912	Jan. 10	7 37
73		5 0	−29	678	J. 1911	Feb. 27	
74		6 0	−16	687	C. 1903	Nov. 20	
75		7 0	− 3	1053	J. 1911	Apr. 1	6 28
76		8 0	+10	1141	J. 1912	Apr. 18	6 12
77		9 0	+22	533	J. 1912	May 6	6 14
78		10 0	+32	527	J. 1911	May 22	6 7
79		11 0	+41	392	J. 1911	May 29	6 2
80		12 0	+47	270	J. 1911	June 23	6 3
81		13 0	+47	100*	J. 1914	May 24	
82		14 0	+44	359	J. 1910	June 28	6 1
83		15 0	+36	508	J. 1910	July 28	5 55
84		16 0	+26	526	J. 1910	June 29	8 20
85		17 0	+14	761	J. 1910	Aug. 4	6 52
86		18 0	+ 2	842	J. 1910	July 11	9 21
87		19 0	−11	1302	J. 1910	Aug. 31	6 57
88		20 0	−24	737	J. 1910	Sept. 2	7 39
89		21 0	−37	577	J. 1910	Sept. 1	8 51
90		22 0	−51	484	J. 1910	Aug. 31	9 27
91		23 0	−63	482	C. 1903	Sept. 25	
92	0	0 0	−61	(394)	J. 1910	Aug. 11	13 13
93		1 0	−62	366	J. 1910	Oct. 6	10 20
94		2 0	−57	222	J. 1910	Sept. 10	13 24
95		3 0	−47	212	J. 1911	Jan. 23	
96		4 0	−35	408	C. 1903	Nov. 21	
97		5 0	−22	292	J. 1911	Feb. 26	
98		6 0	− 9	556	C. 1903	Nov. 27	
99		7 0	+ 4	1132	J. 1911	Mar 28	
100		8 0	+17	763	J. 1911	Apr. 26	6 14
101		9 0	+30	510	J. 1912	May 5	5 53
102		10 0	+43	415	J. 1911	May 23	5 52
103		11 0	+53	(312)	J. 1911	May 28	5 51
104		12 0	+61	298	C. 1904	Mar. 17	
105		13 0	+62	190	J. 1910	May 4	8 32
106		14 0	+57	222	J. 1910	May 9	9 31
107		15 0	+47	229	J. 1910	May 14	9 29
108		16 0	+35	314	J. 1910	June 28	8 30
109		17 0	+22	533	J. 1910	July 31	7 8
110		18 0	+ 9	399	J. 1910	June 2	11 58
111		19 0	− 4	730	J. 1910	Aug. 3	8 20
112		20 0	−17	(700)	J. 1910	Aug. 9	9 8
113		21 0	−30	(635)	J. 1910	Aug. 10	10 26
114		22 0	−43	386	J. 1910	Aug. 11	10 56
115		23 0	−53	194	J. 1910	Sept. 24	9 17
116	+15	0 0	−46	268	M.H. 1908	Oct. 20	
117		1 0	−47	168	M.H. 1907	Dec. 5	
118		2 0	−44	174	M.H. 1908	Oct. 29	

No.	Centre			No. of Thousands of Stars in 225 sq. deg.	Place and Date of Taking (G.M.T. reckoned from Noon)		
	Dec.	R.A.	Gal. Lat.				
	°	h m	°				h m
119	+15	3 0	−36	306	J. 1911	Dec. 14	
120		4 0	−26	172	M.H. 1907	Dec. 5	
121		5 0	−14	427	J. 1911	Feb. 25	
122		6 0	− 2	507	M.H. 1909	Feb. 19	
123		7 0	+11	616	M.H. 1909	Jan. 19	
124		8 0	+24	590	J. 1911	Apr. 22	
125		9 0	+37	368	M.H. 1908	Nov. 25	
126		10 0	+51	182	M.H. 1909	Jan. 16	
127		11 0	+63	58*	M.H. 1908	Apr. 3	
128		12 0	+74	35*	M.H. 1908	Feb. 28	
129		13 0	+77	156	M.H. 1908	May 6	
130		14 0	+68	102	J. 1910	July 4	5 52
131		15 0	+56	415	J. 1910	July 5	6 34
132		16 0	+42	207	M.H. 1908	May 6	
133		17 0	+29	249	M.H. 1908	May 22	
134		18 0	+16	544	M.H. 1908	June 25	
135		19 0	+ 3	(425)	J. 1910	Aug. 5	
136		20 0	−10	1355	J. 1910	July 31	9 37
137		21 0	−22	392	M.H. 1908	Aug. 27	
138		22 0	−32	266	M.H. 1908	Aug. 30	
139		23 0	−41	(176)	J. 1910	July 9	13 59
140	+30	0 0	−32	401	M.H. 1908	Dec. 18	
141		1 0	−33	393	M.H. 1909	Jan. 19	
142		2 0	−30	533	M.H. 1909	Jan. 15	
143		3 0	−24	405	J. 1911	Dec. 15	7 38
144		4 0	−16	(500)	J. 1912	Jan. 18	7 24
145		5 0	− 6	690	M.H. 1909	Jan. 19	
146		6 0	+ 5	869	M.H. 1908	Oct. 20	
147		7 0	+17	662	J. 1911	Mar. 24	
148		8 0	+29	595	M.H. 1909	Jan. 16	
149		9 0	+42	253	M.H. 1909	Feb. 20	
150		10 0	+55	(395)	M.H. 1905	Feb. 11	
151		11 0	+68	105	M.H. 1908	Mar. 23	
152		12 0	+80	28*	M.H. 1908	Jan. 3	
153		13 0	+85	185	M.H. 1909	Feb. 18	
154		14 0	+72	300	M.H. 1908	May 22	
155		15 0	+60	190	M.H. 1908	May 1	
156		16 0	+47	350	J. 1910	June 2	9 48
157		17 0	+34	371	M.H. 1908	May 27	
158		18 0	+22	366	M.H. 1908	June 28	
159		19 0	+10	602	J. 1912	Sept. 2	6 41
160		20 0	− 2	(645)	J. 1912	Sept. 3	7 31
161		21 0	−12	535	M.H. 1907	Sept. 10	
162		22 0	−21	(510)	J. 1912	Sept. 2	8 54
163		23 0	−28	402	J. 1912		8 8
164	+45	0 0	−17	571	M.H. 1908	Nov. 23	
165		1 20	−17	502	M.H. 1908	Nov. 23	
166		2 40	−13	646	M.H. 1908	Dec. 28	
167		4 0	− 5	359	M.H. 1907	Dec. 10	

No.	Centre			No. of Thousands of Stars in 225 sq. deg.	Place and Date of Taking (G.M.T. reckoned from Noon)	
	Dec.	R.A.	Gal. Lat.			
	°	h m	°			h m
168	+45	5 20	+ 6	443	M.H. 1909	Feb. 13
169		6 40	+19	271	M.H. 1908	Dec. 17
170		8 0	+33	199	M.H. 1908	Feb. 23
171		9 20	+46	138	M.H. 1905	Mar. 27
172		10 40	+60	110	M.H. 1908	Apr. 24
173		12 0	+71	65	M.H. 1908	Feb. 1
174		13 20	+71	97	M.H. 1908	Mar. 28
175		14 40	+60	188	M.H. 1908	Apr. 3
176		16 0	+47	194	M.H. 1907	May 10
177		17 20	+33	151	M.H. 1908	June 6
178		18 40	+19	272	M.H. 1908	June 22
179		20 0	+ 7	453	M.H. 1908	July 30
180		21 20	− 4	470	M.H. 1908	July 28
181		22 40	−13	872	M.H. 1908	Oct. 2
182	+60	0 0	− 2	579	M.H. 1908	Nov. 25
183		1 36	− 2	533	M.H. 1908	Nov. 26
184		3 12	+ 3	436	M.H. 1908	Nov. 25
185		4 48	+11	557	M.H. 1909	Feb. 18
186		6 24	+21	415	M.H. 1908	Nov. 25
187		8 0	+33	150	M.H. 1908	Jan. 3
188		9 36	+45	142	M.H. 1908	Mar. 28
189		11 12	+54	147	M.H. 1909	Feb. 21
190		12 48	+57	123	M.H. 1906	Apr. 28
191		14 24	+53	87	M.H. 1908	May 19
192		16 0	+43	83	M.H. 1908	May 23
193		17 36	+32	240	M.H. 1906	June 10
194		19 12	+20	491	M.H. 1908	Oct. 2
195		20 48	+ 9	208	M.H. 1907	Sept. 8
196		22 24	+ 2	486	M.H. 1906	Sept. 24
197	+75	0 0	+13	408	M.H. 1908	Nov. 25
198		2 40	+14	209	M.H. 1908	Dec. 18
199		5 20	+21	316	M.H. 1909	Feb. 12
200		8 0	+31	230	M.H. 1908	Mar. 21
201		10 40	+40	223	M.H. 1908	Dec. 17
202		13 20	+42	232	M.H. 1908	May 26
203		16 0	+36	270	M.H. 1909	Feb. 13
204		18 40	+26	221	M.H. 1908	Aug. 28
205		21 20	+17	305	M.H. 1908	Sept. 21
206	+90	0 0	+27	299	M.H. 1908	Oct. 20